

Recognizing Numerical Constants

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1 Abstract

Abstract The advent of inexpensive, high-performance computers and new efficient algorithms have made possible the automatic recognition of numerically computed constants. In other words, techniques now exist for determining, within certain limits, whether a computed real or complex number can be written as a simple expression involving the classical constants of mathematics.

These techniques will be illustrated by discussing the authors' work in recognizing Euler sums and in finding new formulas for π , π^2 , $\log^2(2)$, formulas that permit digits to be extracted from their expansions.

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2 Introduction

1. Introduction The advent of inexpensive, high-performance computers and new efficient algorithms have made possible the automatic recognition of numerically computed constants. In other words, techniques now exist for determining, within certain limits, whether a computed real or complex number can be written as a simple expression involving the classical constants of mathematics.

The fundamental technique involved is that of finding integer relations. Let $x = (x_1, x_2, \dots, x_n)$ be a vector of real numbers. x is said to possess an integer relation if there exist integers a_i not all zero such that $a_1x_1 + a_2x_2 + \dots + a_nx_n = 0$. By an integer relation algorithm, we mean an algorithm that is guaranteed (provided the computer implementation has sufficient numeric precision) to recover the vector of integers a_i , if it exists, or to produce bounds within which no integer relation can exist.

The problem of finding integer relations among a set of real numbers was first studied by Euclid, who gave an iterative algorithm (the Euclidean algorithm), which when applied to two real numbers, either terminates, yielding an exact relation, or produces an infinite sequence of approximate relations. The generalization of this problem for $n > 2$ has been attempted by Euler, Jacobi, Poincare, Minkowski, Perron, Brun, Bernstein, among others. However, none of their algorithms has been proven to work for $n > 3$, and numerous counterexamples have been found.

The first integer relation algorithm with the desired properties mentioned above was discovered by Ferguson and Forcade in 1977 [14]. In the intervening years a number of other integer relation algorithms have been discovered, including the “LLL” algorithm [16], the “HJLS” algorithm [15], and the “PSOS” [6] algorithm.

3 The PSLQ Integer Relation Algorithm

2. The PSLQ Integer Relation Algorithm In 1991 a new algorithm, known as “PSLQ” algorithm, was developed by Ferguson [12]. It appears to combine some of the best features separately possessed by previous algorithms, including fast run times, numerical stability,

numerical efficiency (i.e. successfully recovering a relation when the input is known to only limited precision), and a guaranteed completion in a polynomially bounded number of iterations. More recently a much simpler formulation of this algorithm was developed, and it has been extended to the complex number field [13]. This newer, simpler version of PSLQ can be stated as follows:

Let x be the n -long input real vector, and let nint denote the nearest integer function (for exact half-integer values, define nint to be the integer with greater absolute value). Let $\gamma := \sqrt{4/3}$. Then perform the following:

Initialize:

1. Set the $n \times n$ matrices A and B to the identity.
2. For $k := 1$ to n : compute $s_k := \sqrt{\sum_{j=k}^n x_j^2}$; endfor. For $k := 1$ to n : $y_k := x_k/s_1$; $s_k := s_k/s_1$; endfor.
3. Compute the $n \times (n - 1)$ matrix H as follows:

For $i := 1$ to n : for $j := i + 1$ to $n - 1$: set $H_{ij} := 0$; endfor; if $i \leq n - 1$ then set $H_{ii} := s_{i+1}/s_i$; for $j := 1$ to $i - 1$: set $H_{ij} := -y_i y_j / (s_j s_{j+1})$; endfor; endfor.

4. Perform full reduction on H , simultaneously updating y , A and B :

For $i := 2$ to n : for $j := i - 1$ to 1 step -1 : $t := \text{nint}(H_{ij}/H_{jj})$; $y_j := y_j + t y_i$; for $k := 1$ to j : $H_{ik} := H_{ik} - t H_{jk}$; endfor; for $k := 1$ to n : $A_{ik} := A_{ik} - t A_{jk}$, $B_{kj} := B_{kj} + t B_{ki}$; endfor; endfor; endfor.

Repeat until precision is exhausted or a relation has been detected:

1. Select m such that $\gamma^i |H_{ii}|$ is maximal when $i = m$.
2. Exchange entries m and $m + 1$ of y , corresponding rows of A and H , and corresponding columns of B .

3. If $m \leq n - 2$ then update H as follows:

Set $t_0 := \sqrt{H_{mm}^2 + H_{m,m+1}^2}$, $t_1 := H_{mm}/t_0$ and $t_2 := H_{m,m+1}/t_0$. Then for $i := m$ to n : $t_3 := H_{im}$; $t_4 := H_{i,m+1}$; $H_{im} := t_1 t_3 + t_2 t_4$; $H_{i,m+1} := -t_2 t_3 + t_1 t_4$; endfor.

4. Perform block reduction on H , simultaneously updating y , A and B :

For $i := m + 1$ to n : for $j := \min(i - 1, m + 1)$ to 1 step -1 : $t := \text{nint}(H_{ij}/H_{jj})$; $y_j := y_j + t y_i$; for $k := 1$ to j : $H_{ik} := H_{ik} - t H_{jk}$; ; endfor; for $k := 1$ to n : $A_{ik} := A_{ik} - t A_{jk}$, $B_{kj} := B_{kj} + t B_{ki}$; endfor; endfor; endfor.

5. Norm bound: Compute $M := 1/\max_j |H_j|$, where H_j denotes the j -th row of H . Then there can exist no relation vector whose Euclidean norm is less than M .

6. Termination test: If the largest entry of A exceeds the level of numeric precision used, then precision is exhausted. If the smallest entry of the y vector is less than the detection threshold, a relation has been detected and is given in the corresponding column of B .

With regards to the termination criteria in step 6, it sometimes happens that a relation is missed at the point of potential detection because the y entry is not quite as small as the detection threshold being used (the threshold is typically set to the “epsilon” of the precision level). When this happens, however, one will note that the ratio of the smallest and largest y vector entries is suddenly very small, provided sufficient numeric precision is being used. In a normal computer run using the PSLQ algorithm, prior to the detection of a relation, this ratio is seldom smaller than 10^{-2} . Thus if this ratio suddenly decreases to a very small value, such as 10^{-20} , then almost certainly a relation has been detected — one need only adjust the detection threshold for the algorithm to terminate properly and output the relation. When detection does occur, this ratio may be thought of as a “confidence level” of the detection.

As a general rule, one can expect to detect a relation of degree n , with coefficients of size 10^m , provided that the input vector is known to somewhat greater than mn digit

precision, and provided that computations are performed using at least this level of numeric precision.

4 Applications of the PSLQ Algorithm

3. Applications of the PSLQ Algorithm There are a number of applications of integer relation detection algorithms in computational mathematics. One application is to analyze whether or not a given constant α , whose value can be computed to high precision, is algebraic of some degree n or less. This can be done by first computing the vector $x = (1, \alpha, \alpha^2, \dots, \alpha^n)$ to high precision and then applying an integer relation algorithm to the vector x . If a relation is found, this integer vector is precisely the set of coefficients of a polynomial satisfied by α . Even if a relation is not found, the resulting bound means that α cannot possibly be the root of a polynomial of degree n , with coefficients of size less than the established bound. Even negative results of this sort are often of interest.

We have performed several computations of this type [6]. These computations have established, for example, that if Euler's constant γ satisfies an integer polynomial of degree 50 or less, then the Euclidean norm of the coefficients must exceed 7×10^{17} . Computations of this sort have also been applied to study a certain conjecture regarding the Riemann zeta function. It is well known [10] that

$$\begin{aligned}\zeta(2) &= 3 \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}} \\ \zeta(3) &= \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} \\ \zeta(4) &= \frac{36}{17} \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}}\end{aligned}$$

These results have led some to suggest that

$$Z_5 = \zeta(5) / \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^5 \binom{2k}{k}}$$

might also be a simple rational or algebraic number. Unfortunately, integer relation calculations [3] have established that if Z_5 satisfies a polynomial of degree 25 or less, then the Euclidean norm of the coefficients must exceed 2×10^{37} .

4. Euler Sums

In response to a letter from Goldbach, Euler considered sums of the form

$$\sum_{k=1}^{\infty} \left(1 + \frac{1}{2^m} + \cdots + \frac{1}{k^m}\right) (k+1)^{-n}.$$

Euler was able to give explicit values for certain of these sums in terms of the Riemann zeta function. For example, Euler found an explicit formula for the case $m = 1, n \geq 2$. Little progress has been made on this problem in the intervening years, although special cases of Euler's results have been rediscovered numerous times (see [7] for some references).

In April 1993, Enrico Au-Yeung, an undergraduate at the University of Waterloo, brought to the attention of one of us the curious fact that

$$\begin{aligned} \sum_{k=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)^2 k^{-2} &= 4.59987\dots \\ &\approx \frac{17}{4} \zeta(4) = \frac{17\pi^4}{360} \end{aligned}$$

based on a computation to 500,000 terms. This author's reaction was to compute the value of this constant to a higher level of precision in order to dispel this conjecture. Surprisingly, a computation to 30 and later to 100 decimal digits still affirmed it.

Intrigued by this empirical result, we computed numerical values for several of these and similar sums, which we have termed Euler sums. We then analyzed these values by a technique we will present below that permits one to determine, with a high level of confidence, whether a numerical value can be expressed as a rational linear combination of several given constants. These efforts produced even more empirical evaluations, suggesting broad patterns and general conjectures. Ultimately proofs were found for many of these experimental results.

We will consider here the following two classes of Euler sums. Some other classes are

considered in [4].

$$\begin{aligned} s_h(m, n) &= \sum_{k=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)^m (k+1)^{-n} & m \geq 1, n \geq 2, \\ s_a(m, n) &= \sum_{k=1}^{\infty} \left(1 - \frac{1}{2} + \cdots + \frac{(-1)^{k+1}}{k}\right)^m (k+1)^{-n} & m \geq 1, n \geq 2, \end{aligned}$$

Explicit evaluations of some of the constants in these classes are presented with proofs in [8] and [9].

5 Numerical Techniques

5. Numerical Techniques It is not easy to compute numerical values of any of these Euler sums to high precision. Straightforward evaluation using the defining formulas, to some upper limit feasible on present-day computers, yields only about eight digits accuracy. Because integer relation algorithms require much higher precision to obtain reliable results, more advanced techniques must be employed.

Our approach to computing numerical values of these sums involves the compound application of the Euler-Maclaurin summation formula (see [2, p. 289]), which can be stated as follows. Suppose $f(t)$ has at least $2p + 2$ continuous derivatives on (a, b) . Let D be the differentiation operator, let B_k denote the k -th Bernoulli number, and let $B_k(\cdot)$ denote the k -th Bernoulli polynomial. Then

$$\begin{aligned} \sum_{j=a}^b f(j) &= \int_a^b f(t) dt + \frac{1}{2}[f(a) + f(b)] \\ &\quad + \sum_{j=1}^p \frac{B_{2j}}{(2j)!} [D^{2j-1} f(b) - D^{2j-1} f(a)] + R_p(a, b). \end{aligned} \tag{1}$$

where the remainder $R_p(a, b)$ is given [2, p. 289] by

$$R_p(a, b) = \frac{-1}{(2p+2)!} \int_a^b B_{2p+2}(t - [t]) D^{2p+2} f(t) dt.$$

We will briefly present a method for computing $s_h(m, n)$. See [4] for more details. Let $h(k) = \sum_{j=1}^k 1/j$ and $f(t) = 1/t$. By the Euler-Maclaurin summation formula,

$$h(k) = \ln k + \frac{1}{2} + \frac{1}{2k} + \sum_{j=1}^p \frac{B_{2j}}{2jk^{2j}} - \sum_{j=1}^p \frac{B_{2j}}{2j} + R_p(1, k).$$

Since $|B_{2k}(t)| \leq |B_{2k}|$ for all k and for $|t| \leq 1$ (see [1, p. 805]), it follows that the remainder $R_p(1, k)$ has a well-defined limit $R_p(1, \infty)$ as k approaches infinity. Now since Euler's constant $\gamma = \lim_{k \rightarrow \infty} [h(k) - \ln k]$, it follows that

$$\begin{aligned} h(k) = & \gamma + \ln k + \frac{1}{2k} - \frac{1}{12k^2} + \frac{1}{120k^4} - \frac{1}{252k^6} + \frac{1}{240k^8} \\ & - \frac{1}{132k^{10}} + \frac{691}{32760k^{12}} - \frac{1}{12k^{14}} + \frac{3617}{8160k^{16}} + O(k^{-18}). \end{aligned} \quad (2)$$

We will use $\bar{h}(k)$ to denote this particular approximation (i.e., (3) without the error term). Now consider the sum

$$s_h(m, n) = \sum_{k=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)^m (k+1)^{-n}.$$

Let c be a large integer, and let $g(t) = \bar{h}^m(t)(t+1)^{-n}$. Applying the Euler-Maclaurin summation formula (1) again, we can write

$$\begin{aligned} s_h(m, n) &= \sum_{k=1}^c \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)^m (k+1)^{-n} \\ &\quad + \sum_{k=c+1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)^m (k+1)^{-n} \\ &= \sum_{k=1}^c h^m(k)(k+1)^{-n} + \int_{c+1}^{\infty} g(t) dt + \frac{1}{2}g(c+1) \\ &\quad - \sum_{k=1}^9 \frac{B_{2k}}{(2k)!} D^{2k-1}g(c+1) + O(c^{-18}). \end{aligned} \quad (3)$$

This formula suggests the following computational scheme. First, explicitly evaluate the sum $\sum_{k=1}^c h^m(k)(k+1)^{-n}$ for $c = 10^8$, using a numeric working precision of 150 digits. Secondly, perform the symbolic integration and differentiation steps indicated in formula (4). Finally, evaluate the resulting expression, again using a working precision of 150 digits. The final result should be equal to $s_h(m, n)$ to approximately 135 significant digits.

The difficulty and cost of performing the symbolic integration and differentiation operations indicated in (4) can be greatly reduced by approximating $g(t)$ as follows: first, expand $\bar{h}^m(t)$, the numerator of $g(t)$, into a sum of individual terms; next, write $(1+t)^{-n}$ as $t^{-n}(1+1/t)^{-n}$; next, expand $(1+1/t)^{-n}$ using the binomial theorem to 18 terms; next, multiply together the resulting numerator and denominator expressions; finally, omit all

terms whose exponent of $1/t$ is greater than 18. The result is a linear sum of terms of the form $t^{-p} \ln^q(t)$ for modest-sized integers p and q .

We have performed many computations of this type. The integration and differentiation operations required in (4) can be handled using a symbolic mathematics package, such as Maple [11] or Mathematica [17]. The explicit summation of the first c terms, as indicated in (4), could be performed by utilizing the multiple precision facility in the Maple or Mathematica packages. However, it was found that the MPFUN multiple precision package and translator developed by one of us [3] was significantly faster for this purpose.

Whatever software is used, this explicit summation is an expensive operation. For example, the evaluation of $s_h(3,4)$ to 10^8 terms, using the MPFUN package with 150-digit precision arithmetic, requires 20 hours on a ‘‘Crimson’’ workstation manufactured by Silicon Graphics, Inc. Thus while such runs can be made, clearly this is pressing the limits of current workstation technology. Fortunately, it is possible to perform such computations on a highly parallel computer system. The details of this parallel algorithm are given in [4].

6 Application of PSLQ to Euler Sums

6. Application of PSLQ to Euler Sums The present application of Euler sum constants is well suited to analysis with integer relation algorithms. We will present but one example of these computations. Consider

$$\begin{aligned} s_a(2,3) &= \sum_{k=1}^{\infty} \left(1 - \frac{1}{2} + \cdots + \frac{(-1)^{k+1}}{k} \right)^2 (k+1)^{-3} \\ &= 0.156166933381176915881035909687988193685776709840 \cdots \end{aligned}$$

Based on experience with other constants, we conjectured that this constant satisfies a relation involving homogeneous combinations of $\zeta(2), \zeta(3), \zeta(4), \zeta(5), \ln(2), \text{Li}_4(1/2)$ and $\text{Li}_5(1/2)$, where $\text{Li}_n(x) = \sum_{k=1}^{\infty} x^k k^{-n}$ denotes the polylogarithm function. The numerical values of these constants, to 50 decimal digits, are as follows:

$$\zeta(2) = 1.644934066848226436472415166646025189218949901206 \cdots$$

$$\begin{aligned}
\zeta(3) &= 1.202056903159594285399738161511449990764986292340 \dots \\
\zeta(4) &= 1.082323233711138191516003696541167902774750951918 \dots \\
\zeta(5) &= 1.036927755143369926331365486457034168057080919501 \dots \\
\ln(2) &= 0.693147180559945309417232121458176568075500134360 \dots \\
\text{Li}_4(1/2) &= 0.517479061673899386330758161898862945622377475141 \dots \\
\text{Li}_5(1/2) &= 0.508400579242268707459108849258589941319541125664 \dots
\end{aligned}$$

The set of terms involving these constants with degree five (see section 7) are as follows: $\text{Li}_5(1/2)$, $\text{Li}_4(1/2) \ln(2)$, $\ln^5(2)$, $\zeta(5)$, $\zeta(4) \ln(2)$, $\zeta(3) \ln^2(2)$, $\zeta(2) \ln^3(2)$, $\zeta(2)\zeta(3)$. When $s_a(2, 3)$ is augmented with this set of terms, all computed to 135 decimal digits accuracy, and the resulting 9-long vector is input to the PSLQ algorithm, it detects the relation $(480, -1920, 0, 16, 255, 660, -840, -160, 360)$ at iteration 390. Solving this relation for $s_a(2, 3)$, we obtain the formula

$$\begin{aligned}
s_a(2, 3) &= 4 \text{Li}_5(1/2) - \frac{1}{30} \ln^5(2) - \frac{17}{32} \zeta(5) - \frac{11}{8} \zeta(4) \ln(2) + \frac{7}{4} \zeta(3) \ln^2(2) \\
&\quad + \frac{1}{3} \zeta(2) \ln^3(2) - \frac{3}{4} \zeta(2) \zeta(3) \\
&= 4 \text{Li}_5(1/2) - \frac{1}{30} \ln^5(2) - \frac{17}{32} \zeta(5) + \frac{11}{720} \pi^4 \ln(2) + \frac{7}{4} \zeta(3) \ln^2(2) \\
&\quad + \frac{1}{18} \pi^2 \ln^3(2) - \frac{3}{24} \pi^2 \zeta(3)
\end{aligned}$$

(recall that $\zeta(2n) = (2\pi)^{2n} |B_{2n}| / [2(2n)!]$).

When the relation is detected, the minimum and maximum y vector entries are 1.60×10^{-134} and 5.98×10^{-29} , respectively. Thus the confidence level of this detection is on the order of 10^{-105} , indicating a very reliable detection.

Although 135-digit input values and 150-digit working precision were used by us when this relation was originally detected, the fact that the maximum y -vector entry is only 10^{-29} at detection implies that such high levels of numeric precision are not required in this case. Indeed, the above relation can be successfully detected using only the 50-digit input values listed above and 50-digit working precision when performing the PSLQ algorithm.

Many special cases of the proven results listed in Table 1 were first obtained using the experimental method presented in sections 2 through 4. In addition, we have obtained a number of experimental results for which formal proofs have not yet been found. Tables 2 and 3 list some of these experimental identities. Others can be found in [4].

It should be emphasized that the results in Tables 2 and 3 are not established in any rigorous mathematical sense by these calculations. However, in each case the “confidence level” (see section 3) of these detections is less than 10^{-50} , and in most cases is in the neighborhood of 10^{-100} .

In many other cases we were not able to obtain a formula for the Euler sum constant explicitly in terms of values of the Riemann zeta, logarithm and polylogarithm functions, but we were able to obtain relations involving two or more Euler sum constants of the same degree (where by “degree” we mean $m+n$, where m and n are the indices of the constant). Some of these relations are shown in Table 3. This is not a complete list; we have obtained numerous other relations of this type. The “confidence level” of each of these relations is smaller than 10^{-25} . The uniqueness of each of these relations was checked by repeating the run with one fewer constant input to PSLQ (there should be no relation detected when this is done).

7 New Formulas for Pi and Related Constants

7. New Formulas for Pi and Related Constants Many readers may be already familiar with the recent paper by the authors and others which gives a new method for computing individual hexadecimal digits of π , as well as a number of other constants. A brief review of these results is as follows. First, let us ask whether π satisfy a relation of the form

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left[\frac{a_0}{8k} + \frac{a_1}{8k+1} + \frac{a_2}{8k+2} + \frac{a_3}{8k+3} + \frac{a_4}{8k+4} + \frac{a_5}{8k+5} + \frac{a_6}{8k+6} + \frac{a_7}{8k+7} \right]$$

$$\begin{aligned}
s_h(3, 2) &= \frac{15}{2}\zeta(5) + \zeta(2)\zeta(3) \\
s_h(3, 3) &= -\frac{33}{16}\zeta(6) + 2\zeta^2(3) \\
s_h(3, 4) &= \frac{119}{16}\zeta(7) - \frac{33}{4}\zeta(3)\zeta(4) + 2\zeta(2)\zeta(5) \\
s_h(3, 6) &= \frac{197}{24}\zeta(9) - \frac{33}{4}\zeta(4)\zeta(5) - \frac{37}{8}\zeta(3)\zeta(6) + \zeta^3(3) + 3\zeta(2)\zeta(7) \\
s_h(4, 3) &= -\frac{109}{8}\zeta(7) + \frac{37}{2}\zeta(3)\zeta(4) - 5\zeta(2)\zeta(5) \\
s_h(5, 4) &= \frac{890}{9}\zeta(9) + 66\zeta(4)\zeta(5) - \frac{4295}{24}\zeta(3)\zeta(6) - 5\zeta^3(3) + \frac{265}{8}\zeta(2)\zeta(7) \\
s_h(6, 3) &= -\frac{3073}{12}\zeta(9) - 243\zeta(4)\zeta(5) + \frac{2097}{4}\zeta(3)\zeta(6) + \frac{67}{3}\zeta^3(3) - \frac{651}{8}\zeta(2)\zeta(7) \\
s_h(7, 2) &= \frac{134701}{36}\zeta(9) + \frac{15697}{8}\zeta(4)\zeta(5) + \frac{29555}{24}\zeta(3)\zeta(6) + 56\zeta^3(3) \\
&\quad + \frac{3287}{4}\zeta(2)\zeta(7) \\
s_a(2, 2) &= 6\text{Li}_4(1/2) + \frac{1}{4}\ln^4(2) - \frac{29}{8}\zeta(4) + \frac{3}{2}\zeta(2)\ln^2(2) \\
s_a(2, 3) &= 4\text{Li}_5(1/2) - \frac{1}{30}\ln^5(2) - \frac{17}{32}\zeta(5) - \frac{11}{8}\zeta(4)\ln(2) + \frac{7}{4}\zeta(3)\ln^2(2) \\
&\quad + \frac{1}{3}\zeta(2)\ln^3(2) - \frac{3}{4}\zeta(2)\zeta(3) \\
s_a(3, 2) &= -24\text{Li}_5(1/2) + 6\ln(2)\text{Li}_4(1/2) + \frac{9}{20}\ln^5(2) + \frac{659}{32}\zeta(5) - \frac{285}{16}\zeta(4)\ln(2) \\
&\quad + \frac{5}{2}\zeta(2)\ln^3(2) + \frac{1}{2}\zeta(2)\zeta(3)
\end{aligned}$$

Table 1: Experimentally Detected Results

$$\begin{aligned}
0 &= 84549s_h(1, 7) + 211468s_h(2, 6) + 148902s_h(3, 5) - 13360s_h(4, 4) - 1978s_h(5, 3) \\
0 &= -2718587s_h(1, 8) - 164525664s_h(2, 7) - 178042944s_h(3, 6) - 88947862s_h(4, 5) \\
&\quad + 3863940s_h(5, 4) + 672100s_h(6, 3) \\
0 &= -14269408s_h(1, 9) + 2578470s_h(2, 8) + 2815376s_h(3, 7) + 5814550s_h(4, 6) \\
&\quad + 6238884s_h(5, 5) + 3938912s_h(6, 4) + 1122784s_h(7, 3) - 1860s_h(8, 2) \\
&\quad + 63164285\zeta(10) \\
0 &= 321\zeta(10) - 440\zeta^2(5) - 720\zeta(3)\zeta(7) - 80\zeta^2(3)\zeta(4) + 560\zeta(2)\zeta(3)\zeta(5) \\
&\quad - 40s_h(2, 8) + 160s_h(3, 7) \\
0 &= -1691755503s_h(1, 10) - 3172589688s_h(2, 9) + 837511504s_h(3, 8) \\
&\quad - 7302717576s_h(4, 7) - 13958660016s_h(5, 6) - 12910466064s_h(6, 5) \\
&\quad - 7099332912s_h(7, 4) - 1773212688s_h(8, 3) + 658360s_h(9, 2) \\
&\quad + 53491434679\zeta(11) - 21868248971\zeta(2)\zeta(9) \\
0 &= -589\zeta(11) + 322\zeta(5)\zeta(6) + 756\zeta(4)\zeta(7) + 254\zeta(3)\zeta(8) - 336\zeta^2(3)\zeta(5) \\
&\quad - 368\zeta(2)\zeta(9) + 80\zeta(2)\zeta^3(3) - 16s_h(3, 8) - 48s_h(4, 7) \\
0 &= 1152s_a(2, 4) + 640s_a(3, 3) - 7680 \ln(2)\text{Li}_5(1/2) + 64 \ln^6(2) - 1881\zeta(6) \\
&\quad + 7440\zeta(5) \ln(2) - 1680\zeta(4) \ln^2(2) - 1120\zeta(3) \ln^3(2) + 864\zeta(3)\zeta(3) \\
&\quad - 640\zeta(2) \ln^4(2) - 432\zeta(2)\zeta(3) \ln(2)
\end{aligned}$$

Table 2: Experimentally Detected Relations

where a_i are rational numbers? Indeed it does. Such a formula can be found by separating the right hand side of the above expression into eight summations, numerically evaluating these to high precision, appending the numerical value of π , and applying PSLQ to the resulting 9-long vector. When this is done, PSLQ discovers the following formula:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left[\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right]$$

A similar formula was discovered by Ferguson shortly after the discovery of the above formula. In fact, there is a two-dimensional lattice of such formulas, which lattice can be generated by these two formulas.

The significance of these formulas for the computation of π can be seen as follows. Let S_1 be the first of the sums in the above formula for π . Then we can write

$$\begin{aligned} \text{frac}(16^d S_1) &= \sum_{k=0}^{\infty} \frac{16^{d-k}}{8k+1} \pmod{1} \\ &= \sum_{k=0}^d \frac{16^{d-k} \pmod{8k+1}}{8k+1} + \sum_{k=d+1}^{\infty} \frac{16^{d-k}}{8k+1} \pmod{1} \end{aligned}$$

The first sum can be rapidly evaluated by means of the binary algorithm for exponentiation, where each operation is performed modulo the integer $8k+1$. These calculations can be done with either integer or floating-point arithmetic, provided the format being used has enough accuracy to exactly represent the integer d^2 . Once an individual exponentiation operation is complete, the resulting integer value is divided by $8k+1$, using floating-point arithmetic, and added to the sum modulo 1. Only a few terms are required of the second, since the terms rapidly become smaller than the “machine epsilon” of the floating-point arithmetic system being used. The resulting fractional value, when expressed in base 16 notation, gives the hexadecimal digits of π beginning at position $d+1$.

Here are a number of other formulas of this type. As before, these formulas were originally found using PSLQ searches.

$$\pi^2 = \frac{1}{8} \sum_{k=0}^{\infty} \frac{1}{64} \left[\frac{144}{(6k+1)^2} - \frac{216}{(6k+2)^2} - \frac{72}{(6k+3)^2} - \frac{54}{(6k+4)^2} + \frac{9}{(k+5)^2} \right]$$

$$\begin{aligned} \pi^2 &= \sum_{k=0}^{\infty} \frac{1}{16^k} \left[\frac{16}{(8k+1)^2} - \frac{16}{(8k+2)^2} - \frac{8}{(8k+3)^2} - \frac{16}{(8k+4)^2} \right. \\ &\quad \left. - \frac{4}{(8k+5)^2} - \frac{4}{(8k+6)^2} + \frac{2}{(8k+7)^2} \right] \\ \log^2(2) &= \frac{-1}{6} \sum_{k=0}^{\infty} \frac{1}{16^k} \left[-\frac{3}{(8k)^2} - \frac{16}{(8k+1)^2} - \frac{40}{(8k+2)^2} - \frac{8}{(8k+3)^2} - \frac{28}{(8k+4)^2} \right. \\ &\quad \left. + \frac{4}{(8k+5)^2} - \frac{28}{(8k+4)^2} - \frac{4}{(8k+5)^2} + \frac{10}{(8k+5)^2} - \frac{2}{(8k+7)^2} \right] \end{aligned}$$

Full details of these calculations, as well as formal proofs of the above formulas, can be found in [5].

8 A General Constant Recognition Procedure

8. A General Constant Recognition Procedure In all of the cases mentioned above, the authors of the respective studies had “hunches” beforehand as to what form the resulting formulas might take. Frankly, some insight of sort is invaluable in avoiding what otherwise is an exponential explosion in the number of possible terms. It simply is not possible to perform integer relation searches with every conceivable term. In fact, if the constant is known to only limited precision, the number of terms that can be considered in an integer relation search may be limited to a handful.

Nonetheless, it does appear feasible to define procedures that are successful in recovering the analytic form of many constants that naturally appear in mathematical calculations. The authors present the following procedure as an example:

1. Using PSLQ and full precision, check if α^j is algebraic of degree n , for j up to m .
2. Using PSLQ and full precision, check if α is given by a multiplicative formula of the form

$$\begin{aligned} 0 &= a_1 \log(\alpha) + a_2 \log(2) + a_3 \log(3) + a_4 \log(5) + \cdots + a_m \log(p_m) \\ &\quad + a_{m+1} \log(c_1) + a_{m+2} \log(c_2) + \cdots + a_{m+n} \log(c_n) \end{aligned}$$

where p_k is the k -th prime, and where c_k are a selected set of transcendentals.

- Using PSLQ and quad precision, check if α is given by a linear formula of the form

$$0 = a_1 + a_2\alpha + a_3t_1 + a_4t_2 + a_5t_3$$

where t_1, t_2 and t_3 are each a product of up to three constants from a set of algebraic and/or transcendental constants.

- If a tentative relation is found in the previous step using quad precision, then check it using full precision.

Some examples of constants recognized by above procedure are the following:

- The root near 1.3851367 of the polynomial

$$\begin{aligned} & -14250992566272 + 1934517374976t^6 - 37548447232t^{12} + 3072863296t^{18} \\ & + 3789095144t^{24} - 408473063t^{30} - 43879700t^{36} - 5815353t^{42} \\ & + 319671t^{48} - 76384t^{54} - 852t^{60} - 444t^{66} + 132t^{72} + 23t^{78} + 3t^{84} + t^{90} \end{aligned}$$

- The definite integral

$$\int_0^\infty t^{7/4} e^{-t} dt = \frac{21\pi\sqrt{2}}{16\Gamma(1/4)}$$

- The definite integral

$$\int_0^{\pi/4} \frac{t^2 dt}{\sin^2(t)} = -\pi^2/16 + \pi \ln(2)/4 + G$$

- The definite integral

$$\int_0^1 \frac{t^2 \ln(t) dt}{(t^2 - 1)(t^4 + 1)} = \frac{\pi^2}{16(2 + \sqrt{2})}$$

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