

Identities inspired from the Ramanujan Notebooks

First series (1998)

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Abstract

I present here a collection of formulas inspired from the Ramanujan Notebooks. These formulas were found using an experimental method based on three widely available symbolic computation programs: PARI-Gp, Maple and Mathematica. A new formula is presented for $\zeta(5)$.

-Introduction

-Results

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The first idea was to find the natural generalization of the $\zeta(3)$ result from Ramanujan. The main problem being to choose among the many possible directions.

$$\zeta(3) = \frac{7\pi^3}{180} - 2 \sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n} - 1)}$$

A first step was to use a variation in the sign. The goal was to get an interesting identity for $\zeta(5)$.

$$\zeta(3) = 14 \sum_{n=1}^{\infty} \frac{1}{n^3 \sinh(\pi n)} - \frac{11}{2} \sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n} - 1)} - \frac{7}{2} \sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n} + 1)}$$

$$\zeta(5) = \frac{\pi^5}{294} - \frac{72}{35} \sum_{n=1}^{\infty} \frac{1}{n^5(e^{2\pi n} - 1)} - \frac{2}{35} \sum_{n=1}^{\infty} \frac{1}{n^5(e^{2\pi n} + 1)}$$

These 2 first formulas found, it was not difficult to guess the next ones.

$$\zeta(5) = -\frac{39}{20} \sum_{n=1}^{\infty} \frac{1}{n^5(e^{2\pi n} - 1)} + \frac{1}{20} \sum_{n=1}^{\infty} \frac{1}{n^5(e^{2\pi n} + 1)} + 12 \sum_{n=1}^{\infty} \frac{1}{n^5 \sinh(\pi n)}$$

Then for $\zeta(7)$

$$\zeta(7) = \frac{19\pi^7}{57600} - 2 \sum_{n=1}^{\infty} \frac{1}{n^7(e^{2\pi n} - 1)}$$

The new formula being the first $\zeta(5)$ formula.

Another direction was suggested by the Apéry formula for $\zeta(3)$.

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3 \binom{2n}{n}}$$

From which the case for $\zeta(5)$ doesn't seem to be there. One possible direction is to use the ψ function and its derivative.

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \binom{2n}{n}} = \frac{\pi\sqrt{3}}{18} \left(\psi' \left(\frac{1}{3} \right) - \psi' \left(\frac{2}{3} \right) \right) - \frac{4}{3} \zeta(3)$$

Which of course leads to $\zeta(5)$ and $\zeta(7)$.

$$\sum_{n=1}^{\infty} \frac{1}{n^5 \binom{2n}{n}} = \frac{\pi\sqrt{3}}{432} \left(\psi''' \left(\frac{1}{3} \right) - \psi''' \left(\frac{2}{3} \right) \right) + \frac{19}{5} \zeta(5) + \frac{\zeta(3)\pi^2}{9}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^7 \binom{2n}{n}} = \frac{11\pi\sqrt{3}}{311040} \left(\psi'''' \left(\frac{1}{3} \right) - \psi'''' \left(\frac{2}{3} \right) \right) - \frac{493}{24} \zeta(7) + \frac{\zeta(5)\pi^2}{9} + \frac{17\zeta(3)\pi^4}{1260}$$

The next step is to find a closed expression for the alternate sums. We know from Apéry's 1979 article that it can be expressed in terms of $\zeta(3)$. If we omit the constant factor for short, we have:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3 \binom{2n}{n}} = [\zeta(3)]$$

But unfortunately, there are no apparent patterns for $\zeta(5)$, $\zeta(7)$, etc. despite my numerous efforts for those sums and previous tests of D. H. Bailey and others are equally inconclusive so far. The mystery remains until someone finds the clue to all this.

An indication for further discoveries in that direction might be given with these remarks:

There are many representations of the same thing, for example,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \binom{2n}{n}} = \left[\psi \left(\frac{1}{5} \right), \psi \left(\frac{2}{5} \right), \psi \left(\frac{3}{5} \right), \psi \left(\frac{4}{5} \right) \right]$$

is in fact a linear combination of these numbers (when simplified).

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \binom{2n}{n}} = [\ln 2\sqrt{5}, \ln(3 + \sqrt{5})\sqrt{5}]$$

That is, a simple linear combination of $\ln(2)\sqrt{5}$ and $\ln(3 + \sqrt{5})\sqrt{5}$. Which is also this number:

$$\frac{2}{5} \operatorname{arcsinh}\left(\frac{1}{2}\right)\sqrt{5}$$

This identity is well known and not difficult to find. So, the case n^5 should be simple and in terms of $\sqrt{5}$ and $\psi'''(\frac{1}{5})$? This is strange since the other formulas were with $\pi\sqrt{3}$ only. It could be also that the $\sqrt{5}$ factor is an artefact. Normally the sum for n^5 should be expressible with $\psi''''(\frac{k}{5}), k = 1..4$ and/or $\psi''''(\frac{1}{3}) \pi\sqrt{3}$ but numerical evidence shows that it is not the case.

Other results

Here are the other identities found with $\zeta(9)$ to $\zeta(21)$ and the results of Joerg Arndt computed on July 27, 1998 up to $\zeta(163)$.

$$\begin{aligned}
 & 37122624 \left[\sum_{n=1}^{\infty} \frac{\left(\frac{1}{n}\right)^9}{n (\exp(2\pi n) - 1)} \right] \\
 & + 74844 \left[\sum_{n=1}^{\infty} \frac{\left(\frac{1}{n}\right)^9}{n (\exp(2\pi n) + 1)} \right] + 18523890 \operatorname{Zeta}(9) - 625 \pi^9 = 0
 \end{aligned}$$

The rest of the listing is here: <http://lacim.uqam.ca/~plouffe/identities.html>

References

- Apéry, Roger Irrationalité de $\zeta(2)$ et $\zeta(3)$, Astérisque 61, 11-13, 1979.
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- van der Poorten, A. "A Proof that Euler Missed... Apéry's Proof of the Irrationality of $\zeta(3)$." Mathematical. Intelligencer 1, 196-203, 1979.
- Zeilberger, D. The Method of Creative Telescoping. Journal of Symbolic Computation 11, 195-204, 1991.
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- Plouffe inverter tables : (2nd collection) : 215 million constants at 16 digits of precision http://pictor.math.uqam.ca/~plouffe/plouffe_inverter_tables/
- Plouffe inverter tables: (3rd collection), 2.853 billion constants (64 digits) : <http://pictor.math.uqam.ca/~plouffe/pi/ip/>