Plouffe's Constant is Transcendental

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Abstract

Plouffe's constant $\frac{\tan^{-1}\left(\frac{1}{2}\right)}{\pi}$ *is transcendental. We prove this and a more general result using sequences of primitive pythagorean triples.*

Plouffe's constant [6], [2],

$$C = \frac{\tan^{-1}\left(\frac{1}{2}\right)}{\pi} \approx 0.14758361765043327417\ldots,$$

is transcendental. In this note, we prove this and the more general result that

Theorem 1

$$\frac{\tan^{-1}(x)}{\pi},$$

is transcendental when x *is rational and* $x \neq 0$ *or* ± 1.

The result is a consequence of the Gelfond-Schneider Theorem which was proved independently by A.O. Gelfond and Th. Schneider in 1934 solving Hilbert's seventh problem from the list of twenty-three outstanding problems that he announced in 1900. The seventh problem dealt with the irrationality and transcendance of certain numbers. One form of the Gelfond-Schneider Theorem is:

Theorem 2 (Gelfond-Schneider) *If* α *and* γ *are non-zero algebraic numbers, and if* $\alpha \neq 1$, *then* $(\log(\gamma))/(\log(\alpha))$ *is either rational or transcendental.*

This is theorem 10.2, p. 135 from Niven, [5]. References are provided in Niven, [5], p. 149.

In the particular case of Plouffe's constant, we take $\alpha = 2/\sqrt{5} + i/\sqrt{5}$, and $\gamma = -1$, then

$$C = \frac{\tan^{-1}\left(\frac{1}{2}\right)}{\pi} = \frac{ln(2/\sqrt{5} + i/\sqrt{5})}{ln(-1)}.$$

More generally, we would have $\alpha = a/\sqrt{a^2+b^2} + bi/\sqrt{a^2+b^2}$, where a and b are relatively prime integers, and

$$\frac{\tan^{-1}\left(\frac{a}{b}\right)}{\pi} = \frac{ln(a/\sqrt{a^2+b^2} + bi/\sqrt{a^2+b^2})}{ln(-1)}.$$

α and γ are algebraic (they are solutions of polynomial equations with integer coefficients), so to show that $\tan^{-1}(a/b)/\pi$ is transcendental, it remains to show that $\tan^{-1}(a/b)/\pi$ is irrational. This result also is given in Niven. By corollary 3.12, [5], p. 41, if $\theta = 2\pi r$ for some rational number r, then $\tan\theta$ is rational only if $\tan\theta = 0, \pm 1$. Niven states this result as a corollary to a theorem due to D.H. Lehmer. We prove this result by other means.

Theorem 3

$$\frac{\tan^{-1}(x)}{\pi},$$

is irrational when x is rational and $x \neq 0$ or ± 1.

Proof: Suppose not, then for some relatively prime positive integers r and s, $\frac{\tan^{-1}(x)}{\pi} = \frac{r}{s}$, and so $\tan^{-1}(x) = \frac{\pi r}{s}$. If this is so, then

$$\tan\left(2s\tan^{-1}\left(\frac{a}{b}\right)\right) = \tan(2\pi r) = 0.$$

We show that no positive integer s with this property exists.

Let $x = \frac{a}{b}$ where a and b are relatively prime integers. Recall the identity:

$$\tan(u+v) = \frac{\tan(u) + \tan(v)}{1 - \tan(u)\tan(v)}. \tag{1}$$

We define the sequence

$$a_n = \tan\left(2n\tan^{-1}\left(\frac{a}{b}\right)\right),$$

which, from equation (1), satisfies the recursion

$$a_n = \frac{a_k + a_{n-k}}{1 - a_k a_{n-k}}$$

for $k = 1, \ldots, n-1$. We can find a_1 by applying equation (1) with $u = v = \tan^{-1}\left(\frac{a}{b}\right)$,

$$\begin{aligned} \tan\left(2\tan^{-1}\left(\frac{a}{b}\right)\right) &= \frac{2\tan\left(\tan^{-1}\left(\frac{a}{b}\right)\right)}{1-\tan\left(\tan^{-1}\left(\frac{a}{b}\right)\right)^2} \\ = a_1 &= \frac{2a/b}{1-\left(\frac{a}{b}\right)^2} \\ &= \frac{2ab}{b^2-a^2}. \end{aligned}$$

Let $p_1 = 2ab$, and $q_1 = b^2 - a^2$ if $b^2 - a^2$ is odd, and let $p_1 = ab$, and $q_1 = (b^2 - a^2)/2$ if $b^2 - a^2$ is even. Then p_1 and q_1 are relatively prime. (If this were not so, then there would be a prime number which divides both ab and hence either a or b, and which divides either $b - a$ or $b + a$. A prime which divides a (b) and $a - b$ must also divide b (a), and similarly a prime which divides a (b) and $a + b$ must divide b (a). But this contradicts our assumption that a and b are relatively prime.)

The absolute values of the numbers $(p_1, q_1, \sqrt{p_1^2 + q_1^2})$ form a primitive pythagorean triple, that is, the numbers $(p_1, q_1, \sqrt{p_1^2 + q_1^2})$ are integers which have no common factor and their absolute values represent sides of a pythagorean triangle.

Ernest Eckert [1] and others [4],[8] have shown that the set of primitive pythagorean triples is a group under the operation called addition defined as follows:

Theorem 4 (Eckert) *The set* $\mathbf{P}$ *of primitive pythagorean triples is a group under the operation called addition, defined by*

$$(a,b,c) + (A,B,C) = \begin{cases} (aA - bB, bA + aB, cC) & \text{when } aA - bB > 0 \\ (bA + aB, bB - aA, cC) & \text{when } aA - bB \leq 0. \end{cases}$$

The identity element in $\mathbf{P}$ *is* $(1,0,1)$, *and the inverse of* (a,b,c) *is* (b,a,c).

¿From this it follows easily that the set of ordered triples of integer components the first two of which may be either positive or negative but whose absolute values form primitive pythagorean triples form a group under addition defined by

$$(a,b,c) + (A,B,C) = (bA + aB, bB - aA, cC),$$

with the identity element $(0,1,1)$ and inverse of (a,b,c) equal to $(-a,b,c)$.

Let $a_n = \frac{p_n}{q_n}$ where p_n and q_n are relatively prime. Then

$$a_{n+1} = \frac{\frac{p_1}{q_1} + \frac{p_n}{q_n}}{1 - \frac{p_1}{q_1}\frac{p_n}{q_n}}$$

$$= \frac{p_1 q_n + p_n q_1}{q_1 q_n - p_1 p_n}.$$

Applying Eckert's theorem,

$$p_{n+1} = p_1 q_n + p_n q_1$$

and

$$q_{n+1} = q_1 q_n - p_1 p_n$$

are relatively prime,

$$\sqrt{p_{n+1}^2 + q_{n+1}^2} = (q_1^2 + p_1^2)^{(n+1)/2}$$

is an integer and $(|p_{n+1}|, |q_{n+1}|, \sqrt{p_{n+1}^2 + q_{n+1}^2})$ is a primitive pythagorean triple. The number p_n cannot be equal to zero for any number n, and hence, no positive integer s exists such that

$$\tan\left(2s \tan^{-1}\left(\frac{a}{b}\right)\right) = \tan(2\pi r) = 0,$$

so

$$\frac{\tan^{-1}(x)}{\pi},$$

is irrational when x is rational and $x \neq 0$ or ± 1. This completes the proof of theorem 3, and this result together with the Gelfond-Schneider theorem proves theorem 1.

As a corollary, we have that

Corollary 1 *The sequences p_n and q_n given by the formulas:*

$$\begin{aligned} p_n &= (a^2+b^2)^n \sin\left(2n \tan^{-1}\left(\frac{a}{b}\right)\right) \\ q_n &= (a^2+b^2)^n \cos\left(2n \tan^{-1}\left(\frac{a}{b}\right)\right). \end{aligned}$$

(for $a \neq 0$, and $a \neq b$, a and b both integers) are integer sequences, and $p_n^2 + q_n^2 = (a^2+b^2)^{2n}$ for all $n = 1, 2, 3, \ldots$, thus

$$(|p_n|, |q_n|, (a^2+b^2)^n)$$

form an infinite sequence of pythagorean triples.

In particular, we have the sequences of ordered triples (sequences A066647, A066648 and A0000351 in EIS[7]):

$$\begin{gathered}(4,3,5),\ (24,-7,5^2),\ (44,-117,5^3),\ (-336,-527,5^4),\ (-3116,-237,5^5),\\ (-10296,11753,5^6),\ (16124,76443,5^7),\ (354144,164833,5^8),\\ (1721764,-922077,5^9),\ (1476984,-9653287,5^{10}),\ (-34182196,-34867797,5^{11}),\\ (-242017776,32125393,5^{12}),\ (-597551756,1064447283,5^{13}),\ \ldots\end{gathered}$$

for the sequence generated by $\tan\left(2n\tan^{-1}\left(\frac{1}{2}\right)\right)$, and

$$\begin{gathered}(12,5,13),\ (120,-119,13^2),\ (-828,-2035,13^3),\ (-28560,-239,13^4),\\ (-145668,341525,13^5),\ (3369960,3455641,13^6),\ (58317492,-23161315,13^7),\\ (13651680,-815616479,13^8),\ (-9719139348,-4241902555,13^9),\\ (-99498527400,95420159401,13^{10}),\ (647549275812,1671083125805,13^{11}),\ \ldots\end{gathered}$$

for the sequence generated by $\tan\left(2n\tan^{-1}\left(\frac{12}{5}\right)\right)$ (sequences A067358, and A067359 in EIS[7]).

Eckert[1], p. 26, also showed that there is one and only one primitive pythagorean triple for each hypotenuse of the form p^n where $p \equiv 1 \mod 4$ is prime and we do not distinguish between triangles (a, b, c) and (b, a, c), so each of these primitive pythagorean triples is unique.

References

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