A One Perturbation Variational Principle and Applications

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Prepared for

Workshop on Well-Posedness

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CECM

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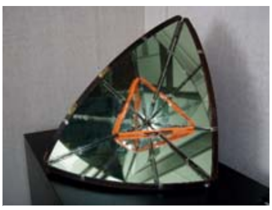
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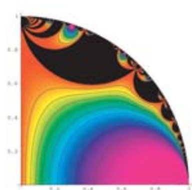
SIMILARITY


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Version of September 2003

- **I am Gordon Shrum Professor of Science at SFU and a Canada Research Chair in IT**

  I am also a Distinguished Fellow of the New Media Innovation Centre
  I belong to CMS, AMS, SIAM, Royal Society of Canada and AAAS

- **I edit for CMS-Springer Books, Proc AMS, Experimental Mathematics**

  Mathematical Methods of OR, Ramanujan J, J. Convex Analysis
  J. Nonlinear & Convex Analysis, Set-valued Analysis, MIA
  SIAM Problems, JIPAM, Applied Nonlinear Analysis

- **My Full Curriculum Vitae is in HTML**

- download CV in Postscript and in Acrobat
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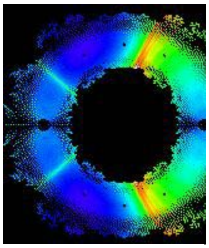

Polynomial roots

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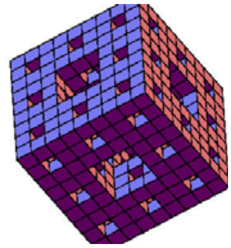
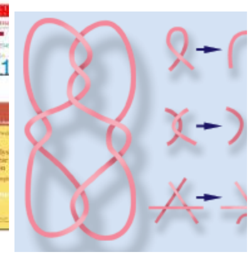


Sierpinski cube (JavaView)

Book collage



Reidemeister moves

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Abstract

We study a variational principle with a *common* perturbation function φ for *all* proper lower semicontinuous extended real-valued functions f on a metric space X.

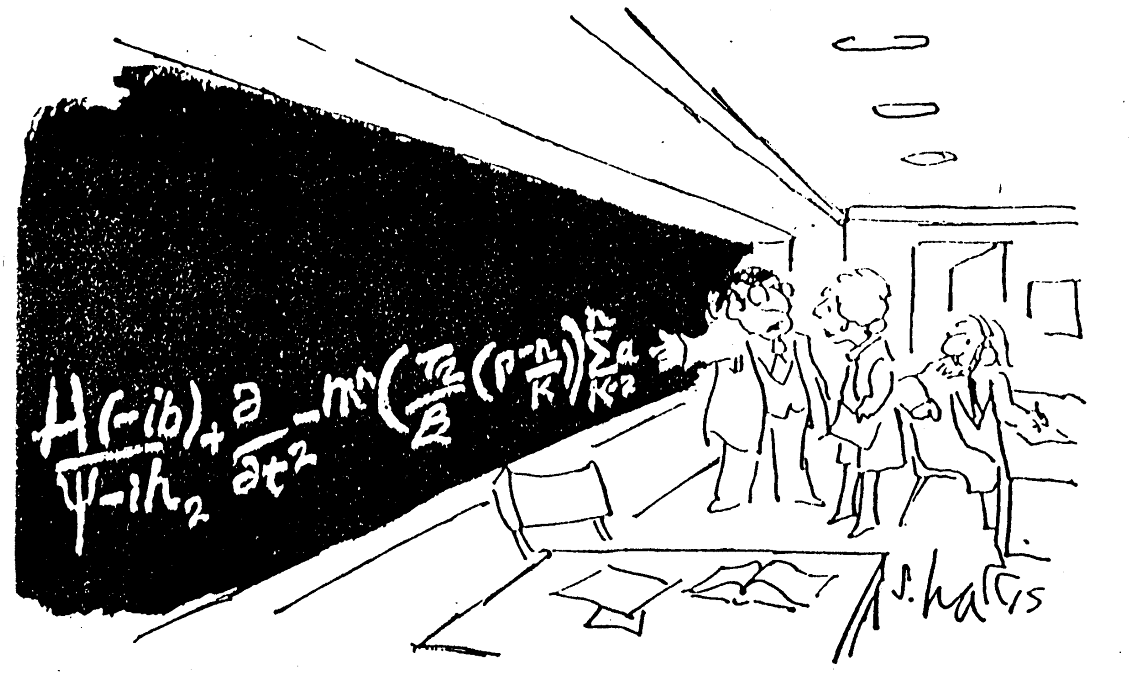
- Necessary and sufficient conditions are given for the perturbed $f+\varphi$ to attain its minimum.

- For separable Banach space we may use a perturbation function that is also convex and *Hadamard-like* differentiable.

- We give applications to differentiability of convex functions on separable and more general Banach spaces.

Credits

- This is NATO sponsored* joint work: J. Borwein, L. Cheng, M. Fabian and J. Revalski, "A one perturbation variational principle with applications," *Set-Valued Analysis*, in press. [CECM Preprint 2003:205]†



"But this *is* the simplified version for the general public."

*Also funded by NSERC, CFI, CRC, etc. !!!

†Available from http://www.cecm.sfu.ca/preprints.

"I'll be glad if I have succeeded in impressing the idea that it is not only pleasant to read at times the works of the old mathematical authors, but this may occasionally be of use for the actual advancement of science."

Constantin Carathéodory (MAA, 1936).

Outline

1. One Theorem

2. Two Applications

3. Three Questions

4. Four References

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Visualization

1. One Theorem

Theorem 1 *Let X be a Hausdorff topological space which admits a proper lsc function*

$$\varphi : X \to \mathbb{R} \cup \{+\infty\}$$

whose level sets are all compact.

Then for any proper lsc and bounded from below function

$$f : X \to \mathbb{R} \cup \{+\infty\}$$

the function $f + \varphi$ attains its minimum.

In particular, if $\operatorname{dom}\varphi$ *is relatively compact, the conclusion is true for any proper lsc function f (not necessarily bounded from below).*

Key step. A function as desired is:

$$\varphi(x) := \begin{cases} \tan\left(\|S^{-1}x\|_H^2\right), & \text{if } \|S^{-1}x\|_H^2 < \frac{\pi}{2}, \\ +\infty, & \text{otherwise.} \end{cases}$$

for an appropriate compact, linear and injective mapping $S\colon H \to X$ $(H := \ell_2)$.

Remark 2 If $(X, \|\cdot\|)$ is *normed* and φ is *convex*, the result above holds for every proper lsc convex f, provided only that the level sets of φ are *weakly* compact, or that $\operatorname{dom}\varphi$ is.

Remark 3 In a normed space $(X, \|\cdot\|)$, by allowing translations of φ, we get a *localization* of the minimum of the perturbation (as in Bishop-Phelps, Ekeland, Borwein-Preiss [B-P], etc.).

With the same proof:

> *Suppose X admits a function φ as above. For any proper lsc (bounded below) function $f : X \to \mathbb{R} \cup \{+\infty\}$, for any $\bar{x} \in \operatorname{dom} f$ and each $\lambda > 0$, the function*
>
> $$f + \varphi((\cdot - \bar{x})/\mu)$$
>
> *(for some $\mu > 0$), attains its minimum at a u with $\|u - \bar{x}\| \leq \lambda$.*

- Observe that in this case, formally, the perturbation function is *now* varying.

- The core requirement of Theorem 1 is also necessary.

Namely, we have:

Theorem 4 *Let* $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ *be a proper function on a metric space* X *with the property that* for every bounded continuous function $f : X \to \mathbb{R}$, the function $f + \varphi$ attains its minimum.

Then φ *is (i) a lower semicontinuous function, (ii) bounded from below, (iii) whose level sets are all compact.*

- This proof is significantly more subtle.

2. Two Applications

1. We recover *Mazur's theorem* (and various *bornological* extensions).

Theorem 5 *Suppose X is a separable Banach space. Then every continuous $f : X \to \mathbb{R}$ is Gâteaux differentiable at the points of a generic (that is a dense G_δ) subset of X.*

Proof. First, we show f is Gâteaux differentiable at the points of a dense subset of X. After translation, it suffices to show every non empty open set Ω of X with $0 \in \Omega$, contains a point at which f is Gâteaux differentiable.

(**Step 1.**) Fix such an Ω and let φ be the function given by Theorem 1. We may suppose $\operatorname{dom}\varphi \subset \Omega$. Then there is an $x \in \operatorname{dom}\varphi \subset \Omega$ at which $-f+\varphi$ attains its minimum.

In particular, for any $h \in \operatorname{dom}\varphi$ and $t > 0$

$$-f(x \pm th) + \varphi(x \pm th) \geq -f(x) + \varphi(x).$$

Using this and convexity of f we obtain

$$\begin{aligned} 0 &\leq f(x+th) + f(x-th) - 2f(x) \\ &\leq \varphi(x+th) + \varphi(x-th) - 2\varphi(x) \end{aligned}$$

which together with the differentiability property (3.1) of φ shows that

$$\lim_{t \searrow 0} \frac{f(x+th) + f(x-th) - 2f(x)}{t} = 0,$$

for every $h \in \operatorname{dom}\varphi$. Since f is locally Lipschitz and $\operatorname{dom}\varphi$ is linearly dense, in fact, the latter limit is 0 for any $h \in X$.

(**Step 2.**) Finally, the fact that f is convex yields its (linear) Gâteaux differentiability at x.

To show the points of Gâteaux differentiability of f is *exactly* a G_δ-subset of X, let us observe that (3.1) yields a stronger conclusion: that X possesses a dense subset in which every x obeys the following stronger condition that as $t \searrow 0$,

$$\sup_{h \in \operatorname{dom} \varphi} \left\{ f(x+th) + f(x-th) - 2f(x) \right\} = o(t). \tag{3.4}$$

On the other hand, the set of all $x \in X$ satisfying (3.4) is always G_δ (possibly empty).

Therefore, f is Hadamard-like, as well as Gâteaux, differentiable on a dense G_δ-subset of X. ©

2. More generally, we recover [C-F, 2001],

$$\text{Sep} \times \text{GDS} \subset \text{GDS}.$$

Proof. Let $f : Y \times X \to \mathbb{R}$ be convex continuous, and $\Omega \subset Y \times X$ be a non empty open set. Assume, for ease, that $2B_Y \times 2B_X \subset \Omega$ and f is bounded on Ω.

(**Step 1.**) Let $\varphi : X \to [0, +\infty]$ be the function provided by Theorem 1 with domain in B_X, and define $g : Y \to (-\infty, +\infty]$ by

$$g(y) := \begin{cases} \inf\{-f(y,x) + \varphi(x);\ x \in X\}, & \text{if } y \in 2B_Y \\ +\infty, & \text{else .} \end{cases}$$

Then g is concave and continuous on $2B_Y$.

As Y is a Gâteaux differentiability space, the function g is Gâteaux differentiable at some y in B_Y.

(**Step 2.**) By Theorem 1, there is $x \in B_X$ so that

$$g(y) = -f(y,x) + \varphi(x).$$

Thus, for every $k \in Y$ and every $h \in \operatorname{dom}\varphi$ we have, for all $t > 0$ sufficiently small,

$$\begin{aligned} & f(y+tk, x+th) + f(y-t\,k, x-t\,h) - 2f(y,x) \\ \leq \;& -g(y+t\,k) + \varphi(x+t\,h) \\ - \;& g(y-t\,k) + \varphi(x-t\,h) + 2g(y) - 2\varphi(x \\ = \;& o(t) + o(t). \end{aligned}$$

Finally, local Lipschitzness of f and linear density of $\operatorname{dom}\varphi$ in X imply

$$f(y+t\,k, x+t\,h) + f(y-t\,k, x-t\,h) - 2f(y,x) = o(t)$$

as $t \searrow 0$, for every $k \in Y$ and every $h \in X$.

Therefore, f is Gâteaux differentiable at the point $(y,x) \in \Omega$. ©

Example 6 (Moors and Somasundaram, 2003)

$$\boxed{\text{WASP} \underset{\neq}{\subset} \text{GDS}}$$

Viscosity is Fundamental

Definition [B-Z, 1996] Let f be lsc and finite at x. We say f is β-*viscosity subdifferentiable* with β-*viscosity subderivative* x^* at x if there is a *locally Lipschitz* g, β-smooth at x, with

$$\nabla^\beta g(x) = x^*$$

and $f - g$ taking a local minimum at x.

We denote the set of all β-viscosity subderivatives of f at x by $\partial_\beta^V f(x)$.

- All variational principles rely *implicitly* or *explicitly* on viscosity subdifferentials.

- We know facts such as ...

- Bornology H = F in Euclidean space.

- Bornology F = WH in reflexive space.

- For locally Lipschitz f

$$\partial_G^V f = \partial_H^V f.$$

- Unless $\ell^1 \subset X$

$$\partial_{WH}^V f = \partial_F^V f$$

for locally Lipschitz convex f.

- When X has a Fréchet renorm

$$\partial_F^V f = \partial_F f$$

(e.g., reflexive or WCG Asplund spaces).

Example 7 *Let $f : \mathbb{R}^n \to \mathbb{R}$ ($n > 1$) be continuous and Gateaux but not Fréchet differentiable at 0. Explicitly in $\mathbb{R}^2$, take*

$$\boxed{f(x,y) := \frac{xy^3}{x^2+y^4}}$$

when $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Let

$$\boxed{g(h) := -|f(h) - f(0) - \nabla_G f(0)h|}$$

Then g is locally uniformly continuous and

(1) Uniquely, $\partial_G g(0) = \{0\}$.

(2) But $\partial_G^V g(0)$ is empty.

Proof. We check that $\nabla_G\, g(0) = 0$, so $\partial_G\, g(0) = \{0\}$. As always

$$\partial_G^V\, g(0) \subset \partial_G g(0).$$

Thus, if (2) fails, $\partial_G^V\, g(0) = \{0\}$, and there is a locally Lipschitz Gateaux-differentiable (hence Fréchet-differentiable) function k such that

$$k(0) = g(0) = 0, \quad \nabla^G\, k(0) = \nabla^G\, g(0) = 0$$

and

$$k \le g$$

in a neighbourhood of 0.

Thus, for small h,

$$\begin{aligned} \frac{|f(0+h) - f(0) - \nabla_G\, f(0)h|}{\|h\|} &\le -\frac{k(h) - k(0)}{\|h\|} \\ &\le \frac{|k(h) - k(0)|}{\|h\|} \end{aligned}$$

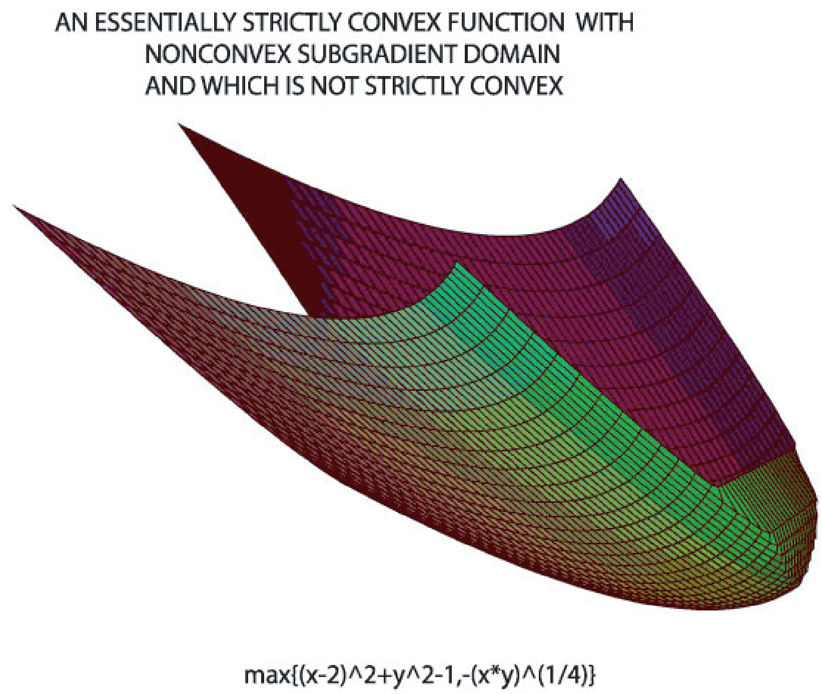
which implies that f is Fréchet-differentiable at 0, a contradiction. ©

2. *Stability.* In light of the (unconditional ZFC) result of *Moors and Somasundaram*, is

$$\text{Sep} \times \text{WASP} \subset \text{WASP}?$$

- Our methods seem only to show that: $\text{Sep} \times \text{a} - \text{WASP} \subset \text{a} - \text{WASP}.$

- For *Stegall's Class*: (G) $\subset$ (S) $\subset$ WASP, and (S) $\times$ (S) $\subset$ (S).



A Legendre-type function

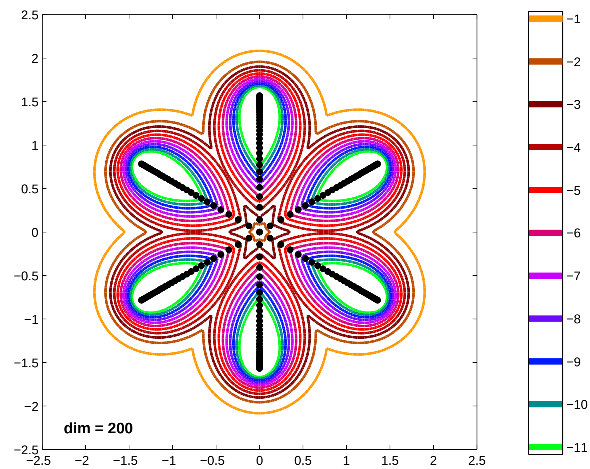
- Drawing often helps ...

3. *Stability.* Linear iterations (needed in work on an amazing continued fraction of *Ramanujan*). Consider

$$\boxed{z_{n+1} = z_n + c_n z_{n-1}, \qquad c_n \to c \in \mathbf{C}}$$

with $z_0 := a, z_0 := b$. *How does its behaviour relate to the case $c_n \equiv c$?*

- What can our field say to address this and other *linear algebra stability* issues?

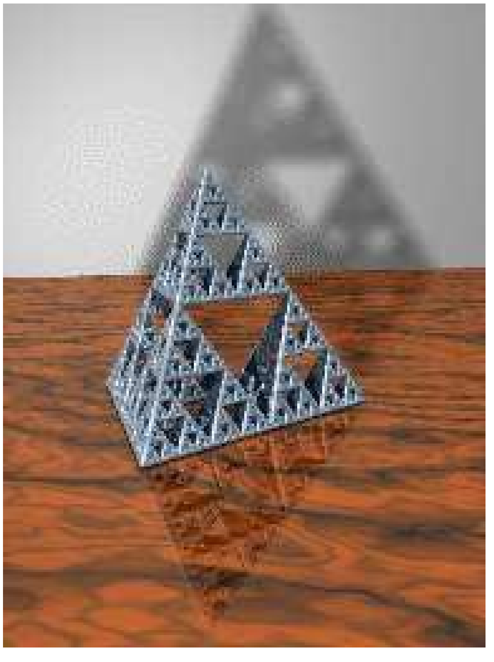


A tri-diagonal pseudospectrum

- What other applications does our principle offer?

Michael Faraday

"The most prominent requisite to a lecturer, though perhaps not really the most important, is a good delivery; for though to all true philosophers science and nature will have charms innumerably in every dress, yet I am sorry to say that the generality of mankind cannot accompany us one short hour unless the path is strewed with flowers."



Virtual

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B-Z: J.M. Borwein and Qiji Zhu, "Viscosity solutions and viscosity subderivatives in smooth Banach spaces with applications to metric regularity," *SIAM J. Optimization*, **34**(1996), 1568–1591.

C-F: L. Cheng, M. Fabian, "The product of a Gâteaux differentiability space and a separable space is a Gâteaux differentiability space," *PAMS* **129**(2001), 3539–3541.

M-S: W. B. Moors and S. Somasundaram, "A Gâteaux differentiability space that is not weak Asplund," *PAMS*, in press.