SERIES OF RECIPROCAL POWERS OF $k-$ALMOST PRIMES

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Abstract. Sums over inverse integer powers s of semiprimes and k-almost primes are reduced to sums over products of powers of ordinary prime zeta functions. Multinomial coefficients known from the cycle decomposition of permutation groups play the role of expansion coefficients. Founded on a known convergence acceleration for the ordinary prime zeta functions, the sums and first derivatives are tabulated with high precision for indices $k = 2, \dots, 6$ and integer powers $s = 2, \dots, 8$.

1. Prime Zeta Function

Definition 1. *The prime zeta function $P(s)$ is the sum over the reciprocal s-th powers of the prime numbers $p = 2, 3, 5, 7, \dots$ [9, 8]*

$$P(s) \equiv \sum_{i=1}^{\infty} \frac{1}{p_i^s}; \quad \Re s > 1. \tag{1}$$

Remark 1. *The primes are represented by sequence A000040 in the Online-Encyclopedia of Integer Sequences [18], and we will adopt the nomenclature that a letter A followed by a 6-digit number names a sequence in this database. Accurate values $P(s)$ for $s \le 9$ are provided by the sequences A085548, A085541, A085964, and A085965–A085969.*

Table 1 complements these by using [17, 7]

$$P(s) = \sum_{p_i \le M} \frac{1}{p_i^s} + \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log P(M, sn) \tag{2}$$

for a suitably large prime M, where μ is the Möbius function [1, 24.3.1], where ζ is the Riemann zeta function [5, 12][10, 9.5], and

$$P(M, s) \equiv \zeta(s) \prod_{p_i \le M} \left(1 - p_i^{-s}\right) \tag{3}$$

an associated definition of a partial product.

The first derivatives $dP(s)/ds$ of (1) are evaluated as the first derivatives of (2) [1, 3.3.6],

$$P'(s) = -\sum_{i=1}^{\infty} \frac{\log p_i}{p_i^s} = -\sum_{p_i \le M} \frac{\log p_i}{p_i^s} + \sum_{n=1}^{\infty} \mu(n) \frac{P'(M, sn)}{P(M, sn)}; \tag{4}$$

$$\frac{P'(M, s)}{P(M, s)} = \frac{\zeta'(s)}{\zeta(s)} + \sum_{p_i \le M} \frac{\log p_i}{p_i^s (1 - p_i^{-s})}. \tag{5}$$

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s	$P(s)$
10	.99360357443698021785585070014773941630187254528520332055356 66(-3) ...
11	.4939472691046549756916217683343987121559397009604952181866074(-3) ...
12	.2460264700345456795266485921650809279799322679473231921741459(-3) ...
13	.1226983675278692799054887924314033239147428525577690135256528(-3) ...
14	.6124439672546447837750803429987454197282126872378013541885303(-4) ...
15	.3058730282327005256755462931371262800130114525389809330765981(-4) ...
16	.1528202621933934418080192641189055977466126987760393110788060(-4) ...
17	.7637139370645897250904556043939762017569839042162662520251345(-5) ...
18	.3817278703174996631227515316311091361624942636382614195748077(-5) ...
19	.1908209076926282572186179987969776618145616195068986381165765(-5) ...
20	.9539611241036233263528834939770057955700555885822134364992986(-6) ...
21	.4769327593684272505083726618818876106041908102543778311286107(-6) ...
22	.2384504458767019281263116852015955086787325069914706736138961(-6) ...
23	.1192199117531882856160246453383398577108304116591413496750467(-6) ...
24	.5960818549833453297113066655008620131146582480117715598724992(-7) ...
25	.2980350262643865876662659401778145949592778827831139923166297(-7) ...
26	.1490155460631457054345907739442373384026574094003717094826319(-7) ...
27	.7450711734323300780164546124093693349559346148927152517177541(-8) ...
28	.3725334010910506351833912287693071753007133176180958755544001(-8) ...
29	.1862659720043574907522145113353601172883347161316571090677067(-8) ...
30	.9313274315523019206770664589654477590951135917359845054142758(-9) ...
31	.4656629062865372188024756168924550748371110904683071460848803(-9) ...
32	.2328311833134403149136721429290134383956012839353792695549588(-9) ...
33	.1164155017134526496600716286019717301900642951759025845555278(-9) ...
34	.5820772087563887361296110329279891461135544371461870778050157(-10) ...
35	.2910385044412396334030528313212481809718543835635609582681388(-10) ...
36	.1455192189083022590216132905087468529073564045445529854681582(-10) ...
37	.7275959835004541439158484817671131286009806802799652884873652(-11) ...
38	.3637979547365416297743239172591421915260411920812352913301657(-11) ...
39	.1818989650303757224685763903905856620025233007531508321634193(-11) ...

TABLE 1. The Prime Zeta Function of some integer arguments. In a style adopted from [1], optional trailing parentheses contain an additional power of 10. Example: The number 3.45×10^{-3} may appear as 0.00345 or as 3.45(−3) or .345(−2). Trailing dots indicate that more digits are chopped off, not rounded, at the rightmost places.

Here, primes denote derivatives with respect to the main argument, which is the second argument for the case of $P(.,.)$. Table 2 shows some of the results for small integer s.

2. Zeta Functions of Almost-primes

2.1. **Nomenclature.** We generalize the notation, and define the k-almost prime zeta functions by summation over inverse powers of k-almost primes,

s	$P'(s)$
2	-4.930911093687644621978262050564912580555881263464682907133271(-1) ...
3	-1.507575555439504221798365163653429195755011615306893318187976(-1) ...
4	-6.060763335077006339223098370971337840638287746125984399112768(-2) ...
5	-2.683860127679835742218751329245015994333014955355822812481980(-2) ...
6	-1.245908072279999152702779277468997004091135047157587587410933(-2) ...
7	-5.940689039148196142550592829016609019368189505929351075166813(-3) ...
8	-2.879524708729247391346028423857334064998983761675865841067618(-3) ...
9	-1.410491921424531291554196456308199977901657131693496192836500(-3) ...
10	-6.956784473446204802000701977708415913844863703329838954712256(-4) ...
11	-3.446864256305149016520798301347221055148509398720732052598028(-4) ...
12	-1.712993524462175657532493112138275372004981118241302276420951(-4) ...
13	-8.530310916711056635208876017215691972617326615054214472499073(-5) ...
14	-4.253630557412291035554757415368617516720893534438843631304558(-5) ...
15	-2.122979056274934599669348621302375720453112762226994727150844(-5) ...
16	-1.060211861676127903320578231686279299852887328732516230264968(-5) ...
17	-5.296802557643848074496697331902062291354582070044729659167083(-6) ...
18	-2.646982787802997352263261854182101806956865359404392741570106(-6) ...
19	-1.323018648512292735443206851957658773372595611301942028763990(-6) ...
20	-6.613517594172600210891457029052100560435779681754585968634604(-7) ...
21	-3.306233614825208657730023089591286331399889373034673500892396(-7) ...
22	-1.652941753425972669328543237067224505237754606957895309294978(-7) ...
23	-8.264125267365738127779160862943622945349018242909935277495509(-8) ...
24	-4.131868136465068742054546598016395808846430264964223880702300(-8) ...
25	-2.065869236367122379085627896761801003317031762395199472960982(-8) ...
26	-1.032913007669833840610060139473968867069681131430230354135307(-8) ...
27	-5.164493003519525183949097124602884025560927707601929381567114(-9) ...
28	-2.582222490193098373680412778703362364263231350157972835761708(-9) ...
29	-1.291103241249637065884459649285079993129747526229207669548247(-9) ...

TABLE 2. The first derivative of the Prime Zeta Function at some integer arguments s.

Definition 2.

$$P_k(s) \equiv \sum_{i=1}^{\infty} \frac{1}{q_i^s}; \quad P_1(s) \equiv P(s). \tag{6}$$

In slight violation of the almost-*terminology, the Prime Zeta Function is incorporated as just one special case,*

$$P_1(s) \equiv P(s). \tag{7}$$

Remark 2. *The sequence q_i is given by the primes A000040 if $k = 1$, by the semiprimes A001358 if $k = 2$, by the 3-almost primes A014612 if $k = 3$, by the 4-almost primes A014613 if $k = 4$, by the 5-almost primes A014614 if $k = 5$, by the 6-almost primes A046306 if $k = 6$, by the 7-almost primes A046308 if $k = 7$ etc.*

Each integer $n > 0$ is either a member of the set $\{1\}$, or of the set of primes, or of the set of semi-primes, etc. These disjoint sets are labeled by the sum of the exponents of the prime number factorizations of their members,

$$\Omega(n) = k; \quad \Omega(1) \equiv 0. \tag{8}$$

The Riemann zeta function may be partitioned into sums over the almost-prime zeta functions,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \sum_{k=1}^{\infty} \sum_{\substack{n=1 \\ \Omega(n)=k}}^{\infty} \frac{1}{n^s} = 1 + \sum_{k=1}^{\infty} P_k(s). \tag{9}$$

Remark 3. *$\zeta(s)$ may be taken from* [1, Table 23.3] *for $s \le 42$, or from A013661, A002117, A013662 – A013678 while $s \le 20$, or from the link to "Recent additions of tables" in Plouffe's database* [15] *while $s \le 99$. $\Omega(n)$ is tabulated in A001222.*

2.2. **Numerical Results.** Table 3 of $P_k(s)$ is deduced from

Theorem 1.

$$P_2(s) = \frac{P(2s) + P^2(s)}{2!}; \tag{10}$$

$$P_3(s) = \frac{2P(3s) + 3P(2s)P(s) + P^3(s)}{3!}; \tag{11}$$

$$P_4(s) = \frac{6P(4s) + 8P(3s)P(s) + 3P^2(2s) + 6P(2s)P^2(s) + P^4(s)}{4!}; \tag{12}$$

$$P_k(s) = \frac{1}{k!} \sum_{\substack{k_1+2k_2+3k_3+\cdots+kk_k=k \\ k_k \ge 0}} (k; k_1 k_2 \ldots k_k)^* P^{k_1}(s) P^{k_2}(2s) \cdots P^{k_k}(ks), \tag{13}$$

utilizing values of the prime zeta function as discussed above and summing over the partitions $\pi(k)$ of k with mixing coefficients

$$(k; k_1 k_2 \ldots k_k)^* = k! / \prod_{m=1}^{k} (m^{k_m} k_m!) \tag{14}$$

of Table 24.2 in [1] *(multinomials M_2) and A036039.*

Proof. (13) is the main result of the paper. For small k, explicit verification can be done along Price's [16] construction, where the k-almost primes fill triangular, ($k = 2$), tetrahedral ($k = 3$) etc. sections of a k-dimensional Euclidean lattice labeled by the prime numbers along its Cartesian axes [2]. The case $P_1(s) = P(s)$ just repeats the definition (6). The case (10) accumulates in $P(2s)$ the sum over the squares, and in $P^2(s)$—with the binomial expansion— again the sum over the squares and twice the sum over products of distinct primes. After division through 2!, each semiprime is effectively represented once.

The generic proof follows through induction: the terms of the right hand side of (13) contain the factor $(k; k_1 k_2, \ldots k_k)^*$, which is the number of distinct permutations with k_m cycles of length m for $m = 1, 2 \ldots k$ [3, p. 123][1, 24.1.2]. The right hand side is the cycle index

$$Z(S_k) \equiv \frac{1}{k!} \sum_{\substack{k_1+2k_2+3k_3+\cdots+kk_k=k \\ k_k \ge 0}} (k; k_1 k_2 \ldots k_k)^* P^{k_1}(s) P^{k_2}(2s) \cdots P^{k_k}(ks) \tag{15}$$

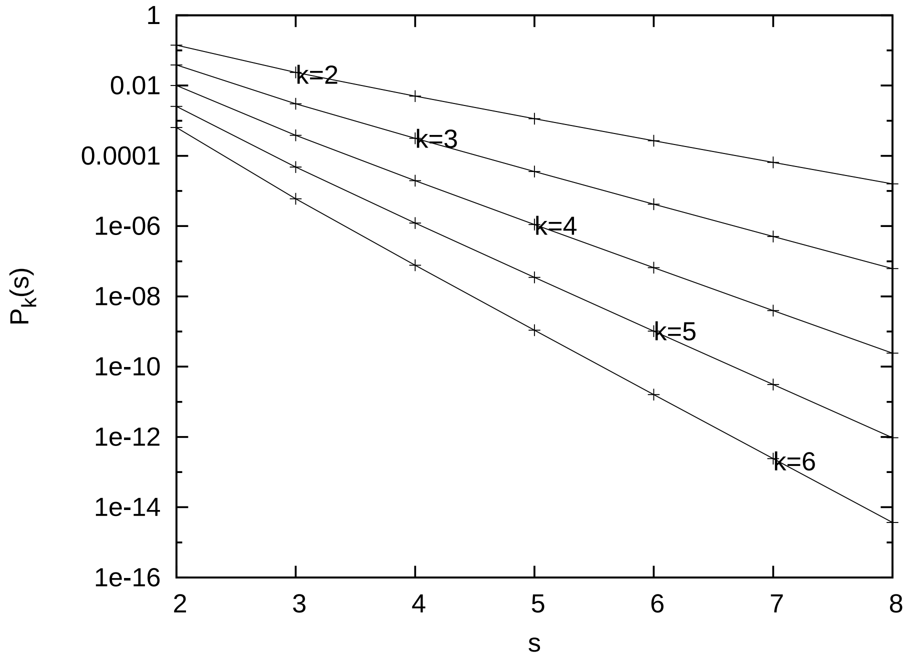

FIGURE 1. A synopsis of table 3 on a semi-logarithmic scale, indicating that the $P_k(s)$ fall off approximately with power laws as $s \to \infty$ along the real s-axis.

of the symmetric group S_k [11, (2.2.5)] with $P(ms)$ substituted for the indeterminates of cycle length m. Skipping any interpretation within a Redfield-Pólya symmetry, its recurrence

$$Z(S_k) = \frac{1}{k} \sum_{j=1}^{k} P(js) Z(S_{k-j}) \tag{16}$$

is already established [11, (2.2.9)]. This matches precisely the recurrence on the left hand side which generates P_k by a combination of products of lower-indexed almost-primes,

$$P_k(s) = \frac{1}{k} \sum_{j=1}^{k} P(js) P_{k-j}(s). \tag{17}$$

This recurrence is valid because each k-almost prime which appears on the left hand side of this equation can be generated in k ways by a product of the form $P(js)P_{k-j}(s)$: in $\omega(k)$ ways by splitting off a prime number and multiplication with a number of the sum in $P_{k-1}(s)$, for each divisor of the k-almost prime which is a square of some prime in addition by multiplication of the square with a term in the sum in $P_{k-2}(s)$, and so on for divisors that are cubes of some prime etc. □

k	s	$P_k(s)$
2	2	1.407604343490233882227509254138772537749192760048802639241489(-1) ...
2	3	2.380603347277195967869595585283620062893217848034845684562765(-2) ...
2	4	4.994674468637339635276874049579289322502057848230867728509096(-3) ...
2	5	1.136012424856354766515556190735772665693748056026108556151424(-3) ...
2	6	2.687071675614096324217387396140875535798787447125719642936101(-4) ...
2	7	6.493314175691145578854061507836714519989975167152833237216508(-5) ...
2	8	1.588851988525958888572372095351879234527858971327233300748108(-5) ...
3	2	3.851619298269464091283792262806039543890016747838157193719155(-2) ...
3	3	3.049362082334312946748098847079302999848694548619577993637287(-3) ...
3	4	3.144274968329417421821246641907192073071706953574340102524412(-4) ...
3	5	3.557725337068269111888017622799305930206716602282958084700573(-5) ...
3	6	4.201275533960671214387834295923202794959720951879928447823823(-6) ...
3	7	5.073887994515979227127878654920650441797124899213497891489656(-7) ...
3	8	6.206813624161469945551964458392656691354524774013736254471908(-8) ...
4	2	1.000943620148325082041084351808525466652473851036634849174401(-2) ...
4	3	3.839045346157269074628008425162843300890790106333110559279434(-4) ...
4	4	1.967963362818191467940855961573410955099879950428233958199339(-5) ...
4	5	1.112105498394147042065416843932409614810339288829649464410277(-6) ...
4	6	6.564866966272364593992630942917565336419279606893244510336952(-8) ...
4	7	3.964020093813893558567748245375642870705531500802782383854435(-9) ...
4	8	2.424542067198129719213221460827573885198415530509018971808441(-10) ...
5	2	2.545076168069302058221776985605516223099431333404435645812102(-3) ...
5	3	4.808940110832567973019045453666287670709263774628437825310148(-5) ...
5	4	1.230321747728495443208363890849979176133316153001252000782537(-6) ...
5	5	3.475459860092756789327837058184938607371038782365590655202548(-8) ...
5	6	1.025765593034930528602801778254441805529589443170682127946500(-9) ...
5	7	3.096892760390829520074774635913281487435638340106655991757766(-11) ...
5	8	9.470868287557099531707292414885674011014480791459591532172493(-13) ...
6	2	6.410338528642807128627067320846767912178898525413482485882042(-4) ...
6	3	6.014928780179108948186295382866155223857573977633433216567001(-6) ...
6	4	7.689936414615761724089452218863183766130186637100554867671046(-8) ...
6	5	1.086086563383684175516301346294503063069865950783005109331817(-9) ...
6	6	1.602759442759730816486790711011122437362717548888087849039669(-11) ...
6	7	2.419447563344257658856721989501609732074582033968956770368419(-13) ...
6	8	3.699557937592796079194223671580493130944594965925089150589532(-15) ...

TABLE 3. Almost-prime zeta functions $P_k(s)$ at small integer arguments s.

Corollary 1. *The product rule* [1, 3.3.3] *applied to (13) yields*

$$P'_k(s) = \frac{1}{k!} \sum_{\substack{k_1+2k_2+3k_3+\cdots+kk_k=k \\ k_k\geq 0}} (k;k_1k_2\ldots k_k)^* P^{k_1}(s)P^{k_2}(2s)\cdots P^{k_k}(ks) \times \left[\frac{k_1P'(s)}{P(s)} + \frac{2k_2P'(2s)}{P(2s)} + \cdots + \frac{kk_kP'(ks)}{P(ks)}\right] \tag{18}$$

k	s	$P_k'(s)$
2	2	-2.8360681540798065222425822254827833607935057823781401341111118(-1) ...
2	3	-3.8805869023993223366924601826587311896741264978521354099521 56(-2) ...
2	4	-7.5458966940853152069079706671963501930216394163596779085014 48(-3) ...
2	5	-1.6552931052406176407610132208680975146293969658668182800471 20(-3) ...
2	6	-3.8397694246350456257553718001194561392116258399832252378936 84(-4) ...
2	7	-9.1747980624691439523464626025702318842641288104202825947601 47(-5) ...
2	8	-2.2297035721814937323522403130216729449868586755076930362229 76(-5) ...
3	2	-1.0927644526886967182339570444603722318742776024289014891094 38(-1) ...
3	3	-7.1768131653384381438718965718681375688595376202629921788618 87(-3) ...
3	4	-6.9571839970163486779987546159176116739083746113625242788174 41(-4) ...
3	5	-7.6592770126954093063747431108080791018988690026393975133120 88(-5) ...
3	6	-8.9189219029602713702850968594500985047258262864582312485260 33(-6) ...
3	7	-1.0687346106887186358836734941326336690942480211039076243534 75(-6) ...
3	8	-1.3012956840596451752212290184488506673670136950785104849043 30(-7) ...
4	2	-3.6037260943517988485066266561811112411308366647969559629328 55(-2) ...
4	3	-1.1741163095729879469778160106188722046408164418223356286444 63(-3) ...
4	4	-5.7229988589129580060173046009023694012892508748977896764203 92(-5) ...
4	5	-3.1655663697960624496652503472309149624354749937765938198758 06(-6) ...
4	6	-1.8487630226185520006114707971908610805356606854134589449442 26(-7) ...
4	7	-1.1097306194195834323077938559497532514943078512458519616900 94(-8) ...
4	8	-6.7637711570592290998115786756788731941641578169651061064895 74(-10) ...
5	2	-1.1021620985050701831041319200537349216582999165216792109774 74(-2) ...
5	3	-1.8061349293879631179892169717077232970411869865884833166215 59(-4) ...
5	4	-4.4313644276805938999208969023370959468110756757017012221370 28(-6) ...
5	5	-1.2302032637587919426962928843124592272277404342776962115949 02(-7) ...
5	6	-3.5997235327081917841845388375229822831208082465108365321617 42(-9) ...
5	7	-1.0816382882901610115099617683225676521588992976051374954335 86(-10) ...
5	8	-3.2985690761637682632747201215210484931167657438350812925303 52(-12) ...
6	2	-3.2327203121505231183040982439695413035428417413560629149531 86(-3) ...
6	3	-2.6769153864443168037448713352769404239823084026861339691441 41(-5) ...
6	4	-3.3028846825906067816988510833674776855023287298323080448213 81(-7) ...
6	5	-4.5972349538834578195270506749575928107611293105024514133272 54(-9) ...
6	6	-6.7355203810850295867808924006008762810997432009159519905140 23(-11) ...
6	7	-1.0127332942097214153313920736275417855604770014344108735085 89(-12) ...
6	8	-1.5449373682949182756961590958029439430665543916623665633885 40(-14) ...

TABLE 4. First derivatives $P_k'(s)$ at small integer arguments s.

as the first derivative of (13), which yields Table 4.

2.3. **Möbius Variant.** Reduction of the summation to k-almost primes with k distinct prime factors defines a signed variant of the prime zeta functions:

Definition 3.

$$P_k^{(\mu)}(s) \equiv \sum_{\substack{i=1 \\ \Omega(q_i)=k}}^{\infty} \frac{\mu(q_i)}{q_i^s} = (-1)^k \sum_{\substack{i=1 \\ \Omega(q_i)=\omega(q_i)=k}}^{\infty} \frac{1}{q_i^s}; \quad P_1^{(\mu)}(s) = -P(s), \tag{19}$$

where $\omega(.)$ is the number of distinct prime factors of its argument.

Remark 4. *The criterion $\Omega(q_i) = \omega(q_i) = k$ selects the products of primes q_i (square-free k-almost primes) of A000040 ($k = 1$), A006881 ($k = 2$), A007304 ($k = 3$), A046386 ($k = 4$), A046387 ($k = 5$), A067885 ($k = 6$), A123321 ($k = 7$), A123322 ($k = 8$), A115343 ($k = 9$), etc.*

Application of the multinomial expansion [1, 24.1.2] to the powers $P^k(s)$ leads to the recurrences

Theorem 2.

$$P_2^{(\mu)}(s) = \frac{P^2(s) - P(2s)}{2!}; \tag{20}$$

$$-P_3^{(\mu)}(s) = \frac{P^3(s) - 3P(s)P(2s) + 2P(3s)}{3!}; \tag{21}$$

$$P_4^{(\mu)}(s) = \frac{P^4(s) - 6P^2(s)P(2s) + 3P^2(2s) + 8P(s)P(3s) - 6P(4s)}{4!}; \tag{22}$$

$$P_k^{(\mu)}(s) = \frac{1}{k!} \sum_{\substack{k_1+2k_2+3k_3+\cdots+kk_k=k \\ k_k \ge 0}} (-1)^m (k; k_1 k_2 \ldots k_k)^* \times P^{k_1}(s) P^{k_2}(2s) \cdots P^{k_k}(ks), \tag{23}$$

where $m \equiv k_1 + k_2 + k_3 + \cdots + k_k$.

Remark 5. *Redistributing the sign with*

$$(-1)^m P^{k_1}(s) P^{k_2}(2s) \cdots P^{k_k}(ks) = P_1^{(\mu)k_1}(s) P_1^{(\mu)k_2}(2s) \cdots P_1^{(\mu)k_k}(ks), \tag{24}$$

shows that a recurrence equivalent to (16) is applicable.

Appendix A. PARI Program Source

A sample program of a PARI/GP interpreter implementation follows [14].

`zetaDerivTkndaux`, `zetaDerivTkNd`, `zetaDerivMcLaux` and `zetaDeriv` implement the computation of $\zeta'(s)$ with the Euler-Maclaurin formula [4, 6, 13]. `zetaLogDeriv` computes ζ'/ζ in (5). `PpartProd` computes $P(M, s)$. `PpartProdDeriv` computes $P'(M, s)/P(M, s)$ in (4). `P` computes $P(s)$ defined in (1). `PDeriv` computes $P'(s)$ in (4). `M2` computes the multinomials of (14). `restPartition` and `partition` compute integer partitions as defining the summation in (13). `Pkalmost` computes $P_k(s)$ with (13). `PkalmostDeriv` computes $P'_k(s)$ with (18). The main program demonstrates construction of the Tables 1–4.

Note that the summation over the partitions in `PpartProd` and `PpartProdDeriv` is implemented with simplicity in mind; it is inefficient since the entire product of prime zeta functions P ought be stored in a look-up table instead of being recomputed from scratch for each new partition of k.

```
/** The d-th derivative w.r.t. s of the product of s+j over j=0 to 2k-2.
* @param[in] k the summation parameter associated with the cut-off parameter M
* @param[in] s the parameter of the zeta function
* @param[in] d the order of the derivative
* @see the term T_{k,n}(s) in eq 29 of Borwein et al, J Comp Appl Math 121 (2000) 247
* @see the term T_{k,n}(s) in eq 1.2 of Odlyzko et al, Trans Am Math Soc 309 (1988) 797
*/
```

```
zetaDerivTkndaux(k,s,d)={
    local(res);
    /* build the product; then calculate the derivative of the polynomial
    */
    res=prod(j=0,2*k-2, x+j) ;
    for(j=1,d,
        res=deriv(res,x)
    ) ;
    /* insert s into the derivative
    */
    substpol(res,x,s) ;
}

/** An auxiliary function of the d-th derivative of the Euler-Maclaurin expansion
* of the zeta function.
* @param[in] k the summation parameter, <= the cut-off parameter M
* @param[in] N the cut-off parameter N
* @param[in] s the argument of the zeta function.
* @param[in] d the order of the derivative
* @return the d-th derivative of T_{k,n}(s) w.r.t. s
* @see the term T_{k,n}(s) in eq 29 of Borwein et al, J Comp Appl Math 121 (2000) 247
* @see the k-sum in eq 6a of Choudhury, Proc R Soc: Math Phys Sci 450 (1995) 477
*/
zetaDerivTkNd(k,N,s,d)={
    local(tkn) ;
    tkn=sum(l=0,d,
        binomial(d,l)*(-1)^l*N^(1-s-2*k)*(log(N))^l
        *zetaDerivTkndaux(k,s,d-l)
    ) ;
    return( bernreal(2*k)/(2*k)!* tkn) ;
}

/** The d-th derivative of N^(1-s)/(s-1).
* @param[in] s the parameter of the zeta function
* @param[in] N the cut-off parameter N in the Euler-McLaurin formula
* @return the value of (d/ds)^d N^(1-s)/(s-1).
* @see the term N^(1-s)/(s-1) in eq 29 of Borwein et al, J Comp Appl Math 121 (2000) 247
*/
zetaDerivMcLaux(s,N,d)={
    /* Use the Leipniz formula for the d-th derivative of
    * the product, the first factor being N^(1-s), and the second
    * being 1/(s-1). Cancel a common factor (-1)^k and a common
    * factor of the binomial(d,k)*k!.
    */
    sum(k=0,d,
        d!/(d-k)!/(s-1)^(k+1) *(-1)^d *N^(1-s)*(log(N))^(d-k)
    );
}

/** The d-th derivative of the Euler-Maclaurin expansion
* of the zeta function.
* @param[in] s the argument of the zeta function.
* @param[in] d the order of the derivative
```

```
* @return the d-th derivative of zeta(s)
* @see eq 29 of Borwein et al, J Comp Appl Math 121 (2000) 247
* @see eq 6a of Choudhury, Proc R Soc: Math Phys Sci 450 (1995) 477
* @see eq 1.2 of Odlyzko et al, Trans Am Math Soc 309 (1988) 797
*/
zetaDeriv(s,d)={
    local(resul,oresul) ;
    /* This is a heuristic scan over the two cutoff parameters
    * M and N, using the Euler-Maclaurin expansion near s=1.
    */
    forstep(M=s+5,1000,5,
        oresul=0.0 ;
        forstep(N=s,1000,5,
            /* Derivatives are summed over the four terms. The
            * first is the sum of n=1 to N-1 of 1/n^s. The second
            * is 1/(2N^s).  The third is N^(1-s)/(s-1).  The fourth
            * is sum over T_{kn}(s) with the Bernoulli factors.
            */
            resul= sum(n=1,N-1,
                (-1)^d*(log(n))^d/n^s) +0.5*(-1)^d*(log(N))^d/N^s
                +zetaDerivMcLaux(s,N,d)
                +sum(k=1,M, zetaDerivTkNd(k,N,s,d) ) ;

            /* Accept a result if the change in the N parameter
            * did not lead to a visible change in the result.
            */
            if( abs( resul-oresul ) < 10.0^(-precision(1.))*abs(resul),
                return(resul) ;
            ) ;
            oresul=resul ;
        ) ;
    ) ;
}

/** Logarithmic derivative of the zeta function.
* @param[in] s argument of the ordinary zeta function
* @return the first logarithmic derivative of the zeta function, zeta'(s)/zeta(s) .
*/
zetaLogDeriv(s)={
    /* The zeta function is part of the PARI standard library. The
    * derivative is implemented via the Euler-Maclaurin formula.
    */
    zetaDeriv(s,1)/zeta(s) ;
}

/** The partial product P(M,s).
* @param[in] s The argument of the zeta function
* @param[in] M The prime number which caps the range of primes in the product
* @return The partial product P(M,s).
*  This is the Zeta function at s, multiplied by the product of 1-p^(-s)
*  over the primes p from 2 up to M.
*/
PpartProd(s,M)={
```

```
    if(! isprime(M),
        error() ;
    ) ;
    zeta(s)*prodeuler(p=2,M, 1-1/p^s) ;
}

/** Logarithmic derivative of the P(M,s) with respect to s.
* @param[in] s The argument of the zeta function
* @param[in] M The prime number which delimits the range of primes in the product
* @return P'(M,s)/P(M,s), the logarithmic derivative of the partial product P(M,s)
*  with respect to s.
*/
PpartProdDeriv(s,M)={
    local(resul) ;
    if(! isprime(M),
        error() ;
    ) ;
    resul=zetaLogDeriv(s) ;
    forprime(p=2,M,
        resul += log(p)/p^s/(1-1/p^s) ;
    ) ;
    return(resul) ;
}

/** Prime zeta function P(s).
* @param[in] s argument. The power in the denominator.
* @return The prime zeta function at s. Sum of 1/p^s over all primes p.
*/
P(s)={
    local(M=nextprime(50),resul=0.) ;
    /* The decomposition with an explicit summation of the primes up to M,
    * and the infinite summation over the Moebius and partial-product terms.
    */
    forprime(p=2,M,
        resul += 1/p^s ;
    ) ;
    resul += suminf(n=1, moebius(n)*log( PpartProd(s*n,M) )/n ) ;
    return(resul) ;
}

/** First derivative of the prime zeta function, P'(s).
* @param[in] s argument of the zeta function
* @return The first derivative of the prime zeta function at s.
*/
PDeriv(s)={
    local(M=nextprime(50),resul=0.) ;
    /* The derivative of the Sebah-Gourdon decomposition with an explicit summation
    * of the primes up to M, and the infinite summation over the Moebius
    * and partial-product terms.
    */
    forprime(p=2,M,
        resul -= log(p)/p^s ;
    ) ;
```

```
    resul += suminf(n=1, moebius(n)* PpartProdDeriv(s*n,M) ) ;
    return(resul) ;
}

/** M2 multinomial factor.
* @param[in] pi a partition of some integer n in the list format [b1,e1,b2,e2,..,bi,ei,...],
*  denoting e1 occurrences of b1, e2 occurrences of b2 etc, n=b1+b1+b1+..+b2+b2+b2+...
* @return the multinomial factor M2 = n!/ product(bi^ei*ei!) .
* @example If pi=[1,2,2,1,3,1], representing 1+1+2+3 =n = 7, the result is
*   420 = 7!/(1^2*2!*2^1*1!*3^1*1!)
* @see Abramowitz and Stegun, Handbook of Mathematical functions, Tab 24.2, chap 24.1.2
*/
M2(pi)={
    local(n=0) ;
    /* The integer n is implicit b1*e1+b2*e2+b3*e3+..
    */
    forstep(i=2, length(pi),2,
        n += pi[i]*pi[i-1] ;
    ) ;
    n = n! ;
    forstep(i=2, length(pi),2,
        n /= pi[i-1]^pi[i] * pi[i]!
    ) ;
    return(n) ;
}

/** Restricted partitions of n
* @param[in] n the number to be decomposed
* @param[in] amin the minimum term in each partition.
* @return a list with members of the form [n1,n2,n3,..], where
* amin<=n1<=n2...<=n, and where the n1+n2+n3+... = n.
* @example n=7 with amin=2 returns [[7],[2,5],[3,4],[2,2,3]]
*/
restPartition(n,amin)={
    local(pi,childp,p) ;
    if( amin<1,
        error("restPartition: expected 2nd argument >=1, found ",amin) ;
    ) ;

    /* the number of partitions here is overestimated.
    */
    pi=listcreate(numbpart(n)) ;

    /* First we create the longer lists with minimum term equal to amin,
    * then the lists with minimum term amin+1 etc up to n itself.
    */
    for(a=amin,n,
        if(a==n,
            /* if the minimum has reached n, the partition is [n]
            */
            childp=listcreate(1) ;
            listput(childp,a) ;
            listput(pi,childp),
```

```
            /* otherwise, we recursively create all lists
            * after peeling off the current value from n, and
            * increasing the minimum term.
            */
            childp=restPartition(n-a,a) ;
            for(l=1,length(childp),
                p=listcreate(1) ;
                listput(p,a) ;
                listput(pi,concat(p,childp[l])) ;
            ) ;
        ) ;
    ) ;
    return(pi) ;
}

/** Integer partitions of n
* @param[in] n the integer to be partitioned.
* @return a list in which each member is a list which shows the
*   partitions and their multiplicity. The format of the members
*   is [b1,e1,b2,e2,b3,e3,...] meaning that the partition contains
*   e1 times b1, e2 times b2 etc, often abbreviated b1^e1 b2^e2 ....
* @example For n =7 the result is similar to
*   [[7,1],[1,1,6,1],[2,1,5,1],[3,1,4,1],[1,2,5,1],[1,1,2,1,4,1],[1,1,3,2],..]
*   representing 7, 1+6, 2+5, 3+4, 1+1+5, 1+2+4, 1+3+3 etc.
*/
partition(n)={
    local(pi,piLong,coL,b,p) ;

    /* first create all unrestricted partitions in the long format
    */
    piLong=restPartition(n,1) ;
    pi=listcreate(length(piLong)) ;

    /* this list is reformatted with a b1^e1, b2^e2 format which
    * shows the "bases" of the terms and their multiplicities, not
    * the terms as often as their multiplicities. Essentially 1+1+1+1
    * is transformed to [1,4], 3+3 to [3,2] etc.
    * Below, col=listcreate(2*A003056(n)) would suffice.
    */
    coL=listcreate(2*n) ;
    for(l=1,length(piLong),
        b=-1 ;
        p=0 ;
        listkill(coL) ;
        for(i=1,length(piLong[l]),
            if( piLong[l][i]==b,
                p++
                ,
                if(p>0,
                    listput(coL,b) ;
                    listput(coL,p) ;
                ) ;
```

```
                b=piLong[l][i] ;
                p=1 ;
            ) ;
        ) ;
        if(p>0,
            listput(coL,b) ;
            listput(coL,p) ;
        ) ;
        listput(pi,coL) ;
    ) ;
    return(pi) ;
}

/** k-almost prime zeta function.
* @param[in] k the bigomega of the k-almost primes q in the sum
* @param[in] s the power in the denominators of the sums
* @return P_k(s), the sum over 1/q^s where q are the k-almost primes.
*/
Pkalmost(k,s)={
    local(pi,resul=0.,m2term,kk,n) ;
    /* create all partitions of k for use of the main theorem
    * of the paper
    */
    pi = partition(k) ;

    /* sum over all partitions */
    for(p=1, length(pi),
        /* pi[p] now points to a partition of the form [b1,e1,b2,e2,...]
        * Multiply the multinomial with the powers of the prime zeta functions
        */
        m2term = M2(pi[p]) ;
        forstep(ki=1, length(pi[p]),2,
            kk= pi[p][ki+1] ;
            n= pi[p][ki] ;
            m2term *= (P(n*s))^kk ;
        ) ;
        resul += m2term ;
    ) ;
    return(resul/k!) ;
}

/** First derivative of k-almost prime zeta function.
* @param[in] k the bigomega of the k-almost primes in the sum
* @param[in] s the power in the denominators of the sums
* @return P'_k(s), the first derivative of P_k(s) with respect to s.
*/
PkalmostDeriv(k,s)={
    local(pi,resul=0.,m2term,kk,n,m2rat) ;
    /* create all partitions of k
    */
    pi = partition(k) ;

    for(p=1, length(pi),
```

```
        /* pi[p] now points to a partition of the form [b1,e1,b2,e2,...].
        * m2term is initialized with  the M2 multinomial of this term.
        */
        m2term = M2(pi[p]) ;
        /* m2rat accumulates the sum over the ratios P'(ns)/P(ns)
        * which is another factor in the sum over all partitions.
        */
        m2rat =0. ;
        forstep(ki=1, length(pi[p]),2,
            /* kk and n mean that there is a component n+n+n+n+..n
            * with kk terms in the partition.
            */
            kk= pi[p][ki+1] ;
            n= pi[p][ki] ;
            m2term *= (P(n*s))^kk ;
            m2rat += n*kk*PDeriv(n*s)/P(n*s) ;
        ) ;
        m2term *= m2rat ;
        /* m2term now concludes the contribution from one of the
        * partitions of k, and is added to the result.
        */
        resul += m2term ;
    ) ;
    return(resul/k!) ;
}

/** Main program: demonstration of the prime zeta function and k-almost-prime
* zeta functions at small integer arguments s.
*/
{
    /* Create a table of P_k(s)
    */
    for(k=2,6,
        for(s=2,8,
            print(k," ",s," ",Pkalmost(k,s) ) ;
        ) ;
    ) ;

    /* Create a table of P'_k(s)
    */
    for(k=2,6,
        for(s=2,8,
            print(k," ",s," ",PkalmostDeriv(k,s) ) ;
        ) ;
    ) ;

    /* Create a table of P(s)
    */
    for(s=10,39,
        print(s," ",P(s)) ;
    ) ;

    /* Create a table of P'(s), the first derivatives
```

```
    */
    for(s=2,29,
        print(s," ",PDeriv(s)) ;
    ) ;
}
```

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